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Cooperative behavior in a model of evolutionary snowdrift games with N-person interactions

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Abstract – We propose a model of evolutionary snowdrift game with N-person interactions and study the effects of multi-person interactions on the emergence of cooperation. An exact N-th–order equation for the equilibrium density of cooperators x^* is derived for a well-mixed population using the approach of replicator dynamics. The results show that the extent of cooperation drops with increasing cost-to-benefit ratio and the number N of interaction persons in a group, with $x^* \sim 1/N$ for large N. An algorithm for numerical simulations is constructed for the model. The simulation results are in good agreement with theoretical results of the replicator dynamics.

The theme of how cooperative behavior emerges among competing entities has attracted the attention of physicists, applied mathematicians, biologists, and social scientists in recent years [1–10]. There are good reasons that physicists showed much interest in this problem and have made contributions. The cooperative behavior is similar to that in interacting spin systems, and some important features, *e.g.*, phase transitions and universality which carry a heavy flavor of statistical physics, have also been observed in evolutionary models of cooperation with spatial structures [9,11]. Indeed, applying ideas in physics across different disciplines is a key characteristic of physics in the new millennium.

A powerful tool to study cooperative phenomena is the theory of evolutionary games based on such basic models as the prisoner's dilemma (PD) [12–14] and the snowdrift game (SG) [15,16]. The basic PD is a two-person game [17,18], in which two players simultaneously choose one of two possible strategies: to cooperate (C) or to defect (D). If one plays C and the other plays D, the cooperator pays a cost $S = -c$ and the defector receives the highest payoff $T = b$ ($b > c > 0$). If both play C, each player receives a payoff $R = b - c > 0$. If both play D, the payoff

is $P = 0$. Thus, the PD is characterized by the ordering $T > R > P > S$ of the payoffs, with $2R > T + S$. In a single encounter, defection is the better action in a well-mixed or fully connected population, regardless of the opponents' decisions. Allowing for repeated encounters and evolution of characters could lead to cooperative behavior [12]. Due to practical difficulties in measuring the payoffs or even ranking the payoffs accurately [19,20], there are serious doubts on taking PD to be the most suitable model for studying emerging cooperative phenomena in a competing setting [21]. The evolutionary snowdrift game (ESG) has been proposed [21] as an alternative to PD and has attracted some recent studies [6–8]. The basic snowdrift game (SG), which is equivalent to the hawk-dove or chicken game [15,16], is again a two-person game. It is most conveniently described using the following scenario. Consider two drivers hurrying home in opposite directions on a road blocked by a snowdrift. Each driver has two possible actions —to shovel the snowdrift (cooperate (C)) or not to do anything (not-to-cooperate or "defect" (D)). If the two drivers cooperate, they could be back home on time and each will get a reward b. Shovelling is a laborious job with a total cost c. Thus, each driver gets a net reward $R = b - c/2$. If both drivers take action D, they both get stuck, and each gets a reward $P = 0$. If only one driver

takes action C and shovels the snowdrift, then both drivers can get through. The driver taking action D (not to shovel) gets home without doing anything and hence gets a payoff $T = b$, while the driver taking action C gets a "sucker" payoff $S = b - c$. The SG refers to the case of $b > c > 0$, leading to $T > R > S > P$. Thus, PD and SG only differ by the order of P and S in the ranking of the payoffs. This seemingly minor difference leads to significant changes in the cooperative behavior, when evolution of characters is introduced. Following replicator dynamics [14], there exists a stable state with coexisting cooperators and defectors in SG for a well-mixed population. More interestingly, it was found that spatial structures tend to suppress the extent of cooperation in ESG [21], in contrast to the common belief that spatial structure constitutes a favorable ingredient for cooperation [22,23].

Most models of evolutionary games proposed so far for studying cooperative phenomena, including those with competitions among a group of entities, involve only two-person interactions. In reality, multi-person interactions are abundant, especially in biological and social systems. A representative model is the so-called public goods game (PGG) [24], for studying group interactions in experimental economics. The PGG considers an interacting group of N agents or players. Each player either contributes a public good of value b at a cost c with $0 < c < b$, or does nothing at no cost. With n cooperators in the group, the total contributions $\mathcal{R}nb$ are divided evenly among all players in the group, where $\mathcal{R}$ ($\mathcal{R} < N$) is called the public good multiplier. Thus a cooperator will get a benefit $\mathcal{R}nb/N - c$, and a defector gets $\mathcal{R}nb/N$ without doing anything. Obviously, in a one-shot PGG, defectors outperform cooperators, leading to a Nash equilibrium where all players are defectors. For $N = 2$, PGG reduces to PD and thus PGG represents a N-person prisoner's dilemma game.

Motivated by the recent works on ESG and PGG, we propose and study a N-person interacting model of SG. We refer to our model as the N-person evolutionary snowdrift game (NESG). The key question is how cooperation is affected by allowing for N-person interactions. The evolution of cooperative behavior in the NESG is studied analytically within the framework of the replicator dynamics [14]. For arbitrary interacting group size N, an exact N-th–order equation for the equilibrium frequency or fraction of cooperators $x^*(r)$ is derived for a well-mixed population, where $r = c/b$ is a parameter that characterizes the cost-to-benefit ratio in SG. The equation can be solved numerically for x^* as a function of r for any N. As the size of the interacting group increases, cooperation in NESG decreases and $x^* \sim 1/N$ for large N. These results are checked against results obtained by numerically simulating the evolutionary dynamics and good agreements are found.

The N-person evolutionary snowdrift game is defined as follows. Consider a system consisting of N_{all} agents. In a N-person game, an agent competes with a group of $N - 1$ other agents. Depending on the situation, the interacting group of N agents can be chosen at random among the N_{all} agents as in the case of a well-mixed population or defined by an underlying geometry as in the case of a regular lattice or other networks. There is a task to be done and every agent will get a reward of b if it is completed by one or more agents within the group. The total cost of performing the task is c, which could be shared among those who are willing to cooperate. The payoff of an agent thus depends on i) the character of the agent and ii) the characters of his $N - 1$ competing agents. Here, we will focus on the case of a well-mixed population.

For an agent of C-character, his payoff depends on the number of C-character agents in the interacting group including himself. The C-character agents are those who are willing to share the labor in completing the task. If the agent under consideration is the sole C-character agent in the group, then his payoff is $b - c$. If there are two C-character agents, then his payoff is $b - c/2$, and so on. Thus, a C-character agent in a N-person snowdrift game has a payoff

$$P_C(n) = b - \frac{c}{n}, \quad n \in [1, N], \tag{1}$$

where n is the number of C-character agents in the group of N agents including the agent concerned.

For an agent of D-character, his payoff depends on whether there is a C-character agent in the group. As long as there is one, the task will be completed and the D-character agent will get a payoff b without doing any work. When there is no C-character agent in the group, then his payoff vanishes since the group has N D-character agents and no one is willing to perform the task. Thus, a D-character agent in a N-person snowdrift game has a payoff

$$P_D(n) = \begin{cases} 0, & n = 0, \\ b, & n \in [1, N-1]. \end{cases} \tag{2}$$

As evolution proceeds in NESG, the numbers of C-character and D-character agents become time-dependent.

The model is original. It is different from the previous models in which the payoffs are typically evaluated by summing up the payoffs of two-player games, for a player competing with a number of other players. There are many real-life situations where pairwise interactions are inapplicable. We give two examples here where N-person interactions are more appropriate. i) In a public construction project such as a bridge, a school or a road that serves a small remote community, everyone in the neighborhood will be benefited (b) and the cost (c) can be shared by those who are willing to contribute. ii) A place such as a class room, a dormitory or a student common room needed to be cleaned regularly with a labor of cost c, and every user will get a benefit b from the cleanliness. Certainly, more realistic modelling will require additional parameters, *e.g.*, more incentives for carrying out the task in the form of long term returns. Here, we study the simplest version as the model can be treated analytically and thus provides

insight into the extent of cooperation with a function of the parameters r and N in the model.

The evolutionary behavior in NESG in a well-mixed population is introduced through the replicator dynamics [14]. The frequency of cooperation $x(t) = N_C(t)/N_{all}$, where $N_C(t)$ is the number of C-character agents in the population at time t [6,21]. The time evolution of $x(t)$ is governed by the following differential equation [14]:

$$\dot{x} = x(f_C - \bar{f}), \qquad (3)$$

where $f_C(t)$ ($\bar{f}$) is the instantaneous average fitness of a C-character agent (the whole population). These quantities are equivalent to the corresponding average payoffs in the case of strong coupling [25]. In the well-mixed case, interacting groups of N agents are randomly chosen. The fitness f_C, which is in general time dependent, is determined as follows according to the binomial sampling [25]

$$f_C = \sum_{j=0}^{N-1} \binom{N-1}{j} x^j (1-x)^{N-1-j} P_C(j+1), \qquad (4)$$

which takes into account the various combinations of the characters of an agent's $N-1$ neighbors. The first three factors in the sum give the probability of having $(j+1)$ C-character agents in the group of N agents. Similarly, the instantaneous average fitness $f_D(t)$ or the average payoff of a D-character agent is given by

$$f_D = \sum_{j=0}^{N-1} \binom{N-1}{j} x^j (1-x)^{N-1-j} P_D(j). \qquad (5)$$

These expressions amount to a mean field approach. In eq. (3), the dynamics of cooperation is that $x(t)$ will increase (decrease) if the fitness f_C is greater (smaller) than the instantaneous average fitness $\bar{f}(t)$ of the whole population. The latter is defined by

$$\bar{f}(t) = x(t) f_C(t) + (1 - x(t)) f_D(t). \qquad (6)$$

Substituting eq. (6) into eq. (3), the dynamics of $x(t)$ is governed by

$$\dot{x} = x(1-x)(f_C - f_D). \qquad (7)$$

Although it is possible to solve the time evolution of $x(t)$, we will instead focus on the steady state. After the transient behavior, the system evolves into a steady state, i.e., the Nash equilibrium, in which $\dot{x} = 0$. It follows from eq. (7) that the steady state or equilibrium frequency of cooperation x^* satisfies

$$f_C(x^*) = f_D(x^*). \qquad (8)$$

Substituting eqs. (1) and (2) into eqs. (4) and (5) gives f_C and f_D in terms of N, b and c. Equation (8) for x^* can then be expressed as

$$\sum_{j=1}^{N-1} \frac{1}{j+1} \binom{N-1}{j} \left(\frac{x^*}{1-x^*} \right)^j = \frac{b-c}{c}. \qquad (9)$$
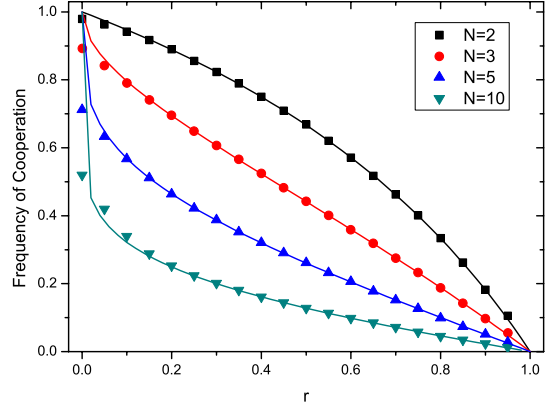

Fig. 1: (Color online) The equilibrium frequency of cooperation as a function of $r = c/b$, for $N = 2, 3, 5$, and 10 in a well-mixed population. The analytic results (lines) obtained by solving eq. (11) and the simulation results (symbols) are in good agreement. In simulations, we used $N_{all} = 2000$ and 10^5 time steps. Each data point is an average over 100 realizations.

Using the identity

$$\sum_{i=0}^{N} \binom{N}{i} \frac{x^{i+1}}{i+1} = \frac{1}{N+1} [(1+x)^{N+1} - 1], \qquad (10)$$

eq. (9) becomes

$$r(1-x^*)^N + N x^* (1-x^*)^{N-1} - r = 0, \qquad (11)$$

which is an N-th-order equation for $x^*(r, N)$ in the steady state, where $r = c/b$. Note that the size of the population N_{all} does not enter, as the analysis assumes an infinite population following the mean-field spirit.

Equation (11) can be solved exactly in closed form for $N \leqslant 4$. For $N = 2$, eq. (11) recovers the result $x^* = \frac{b-c}{b-c/2}$ of the standard two-person evolutionary SG in a well-mixed population [6,21]. For $N \geqslant 5$, eq. (11) can be solved numerically for $x^*(r, N)$. Figure 1 shows the results (lines) of $x^*(r)$ for $N = 2, 3, 5, 10$. We note that $x^*(r)$ decreases as r increases for arbitrary N, with a more rapid drop as r increases for larger values of N. This indicates that the incentives for being a cooperator drops as r and N increase, and agents tend to wait for someone else to perform the task and enjoy a free ride. For a given r, the dependence of x^* on N is shown in fig. 2 on a log-log scale. The results (lines) show that x^* decreases with increasing N, with a power law of exponent -1 for large N. Analytically, the large N behavior can be extracted by taking the small x^* limit of eq. (11). We find

$$x^* = \frac{2(1-r)}{(N-1)(2-r)}, \qquad (12)$$

from which $x^* \sim 1/N$ for large N follows.

As a supplement and to verify the results using the replicator dynamics, we also perform numerical simulations on NESG. The algorithm goes as follows. An agent
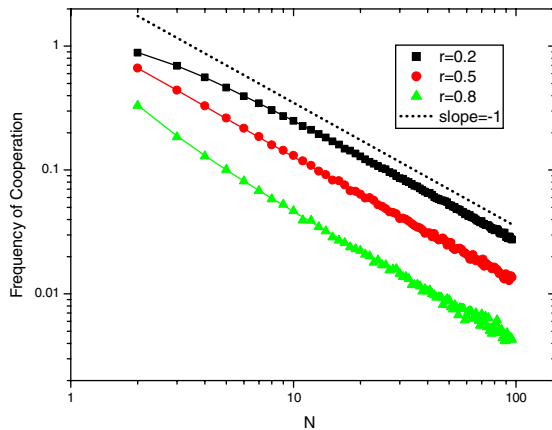
Fig. 2: (Color online) A log-log plot of the equilibrium frequency of cooperation as a function of N for $r = 0.2, 0.5$, and 0.8 in a well-mixed population. The analytic results (lines) from eq. (11) and the simulation results (symbols) are in good agreement. In simulations, we used $N_{all} = 5000$ and 10^7 time steps. Every data point is an average over 10 realizations. A dotted line of slope -1 is shown as a guide to the eye.

in a total population of N_{all} agents can take on either the C-character or D-character. The initial characters of the agents are assigned randomly. At each time step, an agent i is randomly chosen and a group of $N-1$ other agents are randomly chosen among the $N_{all} - 1$ agents to compete with i. Depending on the character of agent i, his payoff P_i is evaluated according to eq. (1) or eq. (2). Evolution of character of agent i is introduced by comparing the payoff with that of another agent j, which is again randomly chosen. For the chosen agent j, he would compete with a randomly chosen group of $N-1$ agents and his payoff is P_j. If P_i is less than P_j, the character of the agent i will be replaced by that of agent j with a probability $(P_j - P_i)/b$. If $P_i \geqslant P_j$, the character of agent i remains unchanged. The results from numerical simulations (symbols in fig. 1 and fig. 2) are in good agreement with the analytic results based on the replicator dynamics. The way of constructing a proper simulation algorithm will also be useful in studying variations of the model in which analytic approaches fail.

In summary, we have proposed and studied an evolutionary snowdrift game with N-person interactions. We derived an exact N-th–order equation for the equilibrium frequency $x^*(r, N)$ of cooperators in a well-mixed population using the approach of replicator dynamics. The results show that the level of cooperation lowers as r increases. For fixed r, x^* drops with the number N of interaction persons in a group and takes on $x^* \sim 1/N$ for large N. We also constructed a numerical algorithm to simulate the model. Simulation data are in good agreement with the analytic results of the replicator dynamics. Further extension of NESG to include the effects of spatial structures such as regular lattices and complex networks will be interesting.

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