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Asymptotic symmetries and integrability: The KdV case

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Abstract – In this letter we consider asymptotic symmetries of the Korteweg de Vries equation, the prototype of the integrable equations. While the reduction of the KdV with respect to point and generalized symmetries gives equations of the Painlevé classification, we show here that the reduction with respect to some asymptotic symmetries violates the Ablowitz-Ramani-Segur conjecture and gives an ordinary differential equation which does not possess the Painlevé property.

Introduction. – According to the Ablowitz, Ramani, Segur (ARS) conjecture [1], all ordinary differential equations (ODE) obtained by an exact symmetry reduction [2] of an integrable partial differential equation have the Painlevé property. The Painlevé property for ODE is defined by looking at the analytic properties of the solution in the complex plane [3,4]. Fixed singularities of the solutions occur at points where the coefficients of the equation are singular while movable singularities of the solutions are those whose location depends just on the initial conditions. The ODE is said to possess the Painlevé property if all its movable singularities are simple poles.

The ARS conjecture also states that the inverse is true, *i.e.* if all the symmetry reductions of a given PDE have the Painlevé property then the PDE should be integrable. Linearizable equations may not have the Painlevé property [5].

Here we consider the reduction of a nonlinear integrable PDE with respect to some of its asymptotic symmetries, *i.e.* when the nonlinear PDE reduces to an ODE in the asymptotic regime of one of its independent variables or in a combination of them, not necessarily to a point at infinity. We will show that in this case the resulting ODE may not have the Painlevé property.

At first we give a brief introduction of the asymptotic symmetries and then we consider the reduction of the Korteweg de Vries equation (KdV) with respect to some of its asymptotic symmetries.

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Asymptotic symmetries. – In most physical applications one is mainly interested in the asymptotic behavior of the solutions of the system and not in the way in which it goes to it. So often, even if the modeling equations are completely integrable and thus we can get explicit exact solutions, these may be complicated and may not provide the information we need. This is also the case of the description of many physical systems given through the KdV. The KdV equation is integrable through a Spectral Transform [6] and we have an infinite class of explicit solutions, the N soliton solutions. The time asymptotic behavior of the generic solution of the KdV is easy to describe as it is just the linear superposition of N solitons corresponding to the N bound states of the initial perturbation as the rest decays asymptotically. These asymptotic solutions can be obtained by solving a simple ODE. This well known result shows that in the asymptotic regime the solution is usually simpler and is described by a simpler equation than the one which governs the behavior of the given nonlinear system for all times and in all space.

So it seems natural to look for solutions just in the asymptotic regime and, as exact solutions are associated to symmetries, to consider the construction of solutions invariant with respect to *asymptotic symmetries*. The notion of asymptotic symmetries has been developed in many articles [7–18] in connection mainly with physical applications. It is strictly related to the application of the renormalization group and singular perturbation theory [14] and it appeared in the work of Penrose [17] when dealing with asymptotic behaviour in special and general relativity. In the study of asymptotic solutions of differential equations with the symmetry methods probably the first coherent presentation has been given by Gaeta in [9] and this is our starting point.

The symmetry method for the study of differential equations, both ordinary or partial differential equations and even difference equations [2,19], has proven in recent years to be one of the most powerful ways to get explicit physically relevant exact solutions for non-linear problems. The Lie symmetries of an equation are obtained in a straightforward way by requiring that the infinitesimal invariance condition be satisfied. Asymptotic symmetries are, in this context, not symmetries of the given system in its whole domain of definition but are symmetries valid only in some restricted region $\mathcal{A}$ of the space $\mathcal{M}$ where the independent variables of our problem are defined. The region $\mathcal{A}$ will usually be a lower dimensional space than $\mathcal{M}$. For asymptotic symmetries the infinitesimal invariance condition will be satisfied only asymptotically, *i.e.* when the independent variables are restricted to the region $\mathcal{A}$. This implies that, given a one parameter symmetry group, we will be just looking for transformations which leave the differential system invariant in $\mathcal{A}$. The definition of the region $\mathcal{A}$ depends on the symmetries we consider.

This definition is better clarified by giving a procedure to look for asymptotic symmetries:

– Choose an infinitesimal transformation depending on a set of arbitrary parameters $\{\alpha_i\}$ and get from it the symmetry variables by integrating the corresponding characteristics.

– Reduce the differential system with respect to the infinitesimal transformation and get the region $\mathcal{A}$ where the reduced differential system is rewritten just in terms of the corresponding symmetry variables. If there exists a region $\mathcal{A}$ and a set of values for the parameters $\{\alpha_i\}$ where this procedure can be carried out then we say that we have an asymptotic symmetry.

As a simple example and to state the notations we will use in the sequel, let us look for some asymptotic symmetries of the heat equation

$$u_t - u_{xx} = 0. \tag{1}$$

Dilation transformations. The vector field

$$X = x\partial_x + 2t\partial_t,$$

provides a dilation symmetry [2] of the heat equation (1). We modify this vector field introducing a parameter α as follows:

$$X(\alpha) = x\partial_x + \alpha t\partial_t, \tag{2}$$

The similarity variable is

$$\xi = \frac{x^\alpha}{t}. \tag{3}$$

Inserting the similarity variable (3) into the heat equation, we get

$$\alpha^2 v'' + [\alpha(\alpha-1)\xi^{-1} + \xi^{-1+2/\alpha} t^{-1+2/\alpha}]v' = 0. \tag{4}$$

This equation contains t, although the function $v(\xi)$ depends only on ξ. The dependence on t in eq. (4) is removed if $\alpha = 2$. In this case we have an exact symmetry reduction of the heat equation:

$$4v'' + (2\xi^{-1} + 1)v' = 0.$$

The corresponding solution of the heat equation is

$$u(x,t) = \int_0^{\frac{x}{\sqrt{2t}}} \mathrm{e}^{-z^2}\mathrm{d}z.$$

We can also consider eq. (4) in the limit $t \to \infty$. In this case, if $\alpha > 2$, we get the ordinary differential equation

$$v'' + \frac{\alpha - 1}{\alpha\,\xi} v' = 0,$$

whose solution is given by

$$v(\xi) = C_1 \xi^{1/\alpha} + C_2.$$

Consequently, we can conclude that

$$u(x,t) = \frac{x}{t^{1/\alpha}} \tag{5}$$

is an explicit asymptotic solution of the heat equation when t goes to infinity with the asymptotic symmetry (2).

Galilean transformations. The vector field

$$X = 2t\partial_x - xu\partial_u,$$

provides another symmetry [2] of the heat equation. The modified vector field is

$$X(\alpha) = 2t\partial_x + \alpha x u\partial_u.$$

The similarity variables associated to this vector field are

$$y = t, \quad v = u\mathrm{e}^{-\alpha x^2/4t}. \tag{6}$$

If we substitute (6) in the heat equation (1), with $v = v(y)$, we obtain

$$v' - \frac{\alpha}{2y}\left(1 + (\alpha+1)\frac{x^2}{2y}\right) v = 0. \tag{7}$$

We get the usual result [2] $v = v_0/\sqrt{y}$ when $\alpha = -1$. For arbitrary α, we cannot obtain an acceptable equation to be considered as an asymptotic limit of the heat equation. When x goes to infinity, both the similarity variables (6) and the resulting eq. (7) are meaningless. We can however consider the limit $x \to 0$, which gives an asymptotic solution $v = v_0 y^{-\alpha/2}$ valid for any α, different from the exact solution corresponding to $\alpha = -1$.

KdV symmetries. – We will apply the previous ideas to the KdV equation

$$u_t - u_{xxx} + 6uu_x = 0. \tag{8}$$

The classification of the group-invariant solutions of the KdV equation is very well known (see [2] for the details). The obtained symmetries are given by scaling: $D = x\partial_x + 3t\partial_t - 2u\partial_u$, Galilean boosts: $B = 6t\partial_x + \partial_u$, time translations: $T = \partial_t$, space translations: $S = \partial_x$ and their combinations.

The scaling invariant solution, $u = t^{-2/3}v$, $y = xt^{-1/3}$, is written in terms of the second Painlevé transcendent, solution of the nonlinear equation

$$w_{yy} = 2w^3 - \frac{1}{3}yw + k, \tag{9}$$

where $v = \frac{\mathrm{d}w}{\mathrm{d}y} + w^2$. The Galilean boost invariant solution is given by $u = \frac{x+k}{6t}$, the time translation invariant solution is given in terms of the Weierstrass function,

$$u = \wp\left(\frac{x}{\sqrt{2}} + c; g_2, g_3\right),$$

and the space translation invariant solution is a constant. We can write solutions invariant with respect to a combination of the Galilean boost and time translation, $B+T$. In this case the invariant solution $u = v + t$, $y = x - 3t^2$, is written in terms of the first Painlevé transcendent, solution of the nonlinear equation

$$v_{yy} - 3v^2 + y + k = 0.$$

In the following we will consider at first asymptotic symmetries of dilation type which provide solutions described by linear ODEs. Then by considering a rotation-like symmetry we get a reduced nonlinear equation which has NOT the Painlevé property.

Dilations. The first vector field we will consider is a scale transformation, where α_1 and α_2 are two arbitrary parameters:

$$X = \alpha_1 t\partial_t + x\partial_x + \alpha_2 u\partial_u.$$

Their invariants are

$$\xi = \frac{x}{t^{1/\alpha_1}}, \qquad u = t^{\alpha_2/\alpha_1}F(\xi).$$

Substituting u into the KdV equation (8), we get

$$F'''(\xi) - 6F(\xi)F'(\xi)t^{(\alpha_2+2)/\alpha_1} + \frac{1}{\alpha_1}\left[\xi F'(\xi) - \alpha_2 F(\xi)\right]t^{(3-\alpha_1)/\alpha_1} = 0. \tag{10}$$

For finite t the equation is inconsistent, since F depends only on ξ, unless $\alpha_1 = 3$, $\alpha_2 = -2$, which corresponds to the well-known dilation symmetry of the KdV equation [2]. In this case eq. (10) is consistent as it does not depend on t,

$$F''' - 6FF' + \frac{\xi}{3}F' + \frac{2}{3}F = 0,$$

and it reduces to the second Painlevé equation (9). An exact solution of the KdV equation in this case is given by $u = \frac{x}{6t}$ corresponding to $F(\xi) = \frac{1}{6}\xi$.

For arbitrary values of α_1, α_2, we get a compatible equation only if we can eliminate the terms containing t, for example by choosing t going to infinity. In this case, we need the exponents of t to be negative. This means

$$\frac{3-\alpha_1}{\alpha_1} \leqslant 0, \qquad \frac{\alpha_2+2}{\alpha_1} \leqslant 0,$$

with solutions

$$\alpha_1 \geqslant 3, \qquad \alpha_2 \leqslant -2, \qquad \text{or} \qquad \alpha_1 < 0, \qquad \alpha_2 \geqslant -2. \tag{11}$$

The possible cases are

a) When α_1 and α_2 satisfy any of the inequalities (11), with $\alpha_1 \neq 3$ and $\alpha_2 \neq -2$, the last two terms of equation (10) vanish in the limit $t \to \infty$ and the resulting equation is simply

$$F''' = 0,$$

with solution

$$F(\xi) = c_1 + c_2\xi + c_3\xi^2.$$

Then, the solution for u is given by

$$u = c_1 t^{\alpha_2/\alpha_1} + c_2 x t^{(\alpha_2-1)/\alpha_1} + c_3 x^2 t^{(\alpha_2-2)/\alpha_1}.$$

b) If $\alpha_1 = 3$ and $\alpha_2 < -2$, eq. (10) (in the limit $t \to \infty$) is

$$F''' + \frac{\xi}{3}F' - \frac{\alpha_2}{3}F = 0,$$

which is also a linear equation. The solution of this equation is written in terms of a combination of hypergeometric functions of argument $-\frac{1}{27}\xi^3$.

c) If $\alpha_1 > 3$ or $\alpha_1 < 0$, and $\alpha_2 = -2$, the equation (in the limit $t \to \infty$) is

$$F''' - 6FF' = 0.$$

This is the same equation one can find using the time translation invariance. The solution is written in terms of a Weierstrass elliptic function $\wp(\xi/\sqrt{2}; g_2, g_3)$.

If $\alpha_1 = 0$ then t is an invariant variable and the other invariant is $u = x^{\alpha_2}G(t)$. The only nontrivial result in this case is obtained when $\alpha_2 = 1$. In this case the asymptotic equation (when $x \to \infty$) for G reads

$$G'(t) + 6G^2(t) = 0,$$

and the general solution is

$$u(x,t) = \frac{x}{6(t-t_0)}.$$

Note that all these asymptotic solutions, obtained by a method close to the reduction method for true symmetries, can be expressed in terms of known functions, satisfying equations sharing the Painlevé property.

Rotations. To get a nontrivial nonlinear equation, let us consider a space-time rotation plus a dilation of the dependent variable. In this case, the vector field is

$$X = x\partial_t - t\partial_x + \alpha u\partial_u. \tag{12}$$

If we integrate eq. (12), the invariants are

$$\rho = \sqrt{x^2+t^2}, \qquad \theta = \arctan\frac{t}{x}, \qquad u = F(\rho)\mathrm{e}^{\alpha\theta}.$$

Writing the KdV equation in terms of these variables we get

$$\begin{aligned}&F''' + \frac{3\tan\theta}{\rho}(\tan\theta - \alpha)F'' + \frac{6\alpha}{\rho}e^{\alpha\theta}(\tan\theta\sec^2\theta)F^2\\&+\left(\frac{1}{2\rho^2}[9\alpha\cos 2\theta + 3(\alpha^2-1)\sin 2\theta - 2\rho^2 + 3\alpha]\tan\theta\right.\\&\left.-6e^{\alpha\theta}F\right)(\sec^2\theta)F' + \frac{\alpha\sec^3\theta}{4\rho^3}[(\alpha^2-8)\sin 3\theta\\&+6\alpha\cos 3\theta - 3\alpha^2\sin\theta - 2(2\rho^2+3\alpha)\cos\theta]F = 0.\end{aligned}$$

The only mathematically meaningful limit which allows us to get a consistent equation for $F(\rho)$ is obtained when $t \to 0$. In this case we get

$$F''' - \left(6F' + \frac{\alpha}{\rho}\right)F = 0, \tag{13}$$

which exists independently of the value of α. Equation (13) has no symmetries, unless $\alpha = 0$.

A Painlevé analysis of eq. (13) using the Baldwin-Hereman program [20] shows that, unless $\alpha = 0$, eq. (13) has not the Painlevé property.

In the case $\alpha = 0$, the resulting equation

$$F''' - 6F'F = 0, \tag{14}$$

can be easily integrated as it has a two-parameter group of symmetries, translations and dilations. The solution of eq. (14) is an elliptic function and it passes the Painlevé test.

Conclusions. – In this work we have considered the asymptotic symmetries for the KdV equation. As is well known all symmetry reductions of KdV are given by ordinary differential equations belonging to the Painlevé class. Here we have shown that the reduction by asymptotic symmetries in some cases can give ordinary differential equations which are not in the Painlevé class. This opens a new perspective in the study of the solutions of integrable systems, as we may not always have analytic solutions in the asymptotic regime. A more detailed analysis of this problem may provide new interesting results in the relation between integrable partial differential equations and equations of Painlevé type.

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REFERENCES

[1] Ablowitz M. J., Ramani A. and Segur H., *Lett. Nuovo Cimento*, **23** (1978) 333.
[2] Olver P., *Applications of Lie Groups to Differential Equations* (Springer Verlag, New York) 1993.
[3] Conte R. (Editor), *The Painlevé Property One Century Later* (Springer Verlag, New York) 1999.
[4] Kowalewskaya S., *Acta Math.*, **14** (1890) 81.
[5] Tamizhmani K. M., Grammaticos B. and Ramani A., *SIGMA*, **3** (2007) 073.
[6] Calogero F. and Degasperis A., *Spectral Transform and Soliton: Tools to Solve and Investigate Nonlinear Evolution Equations*, Vol. **1** (North-Holland Publishing Company, Amsterdam) 1982.
[7] Bricmont J. and Kupiainen A., *Commun. Math. Phys*, **150** (1992) 193.
[8] Chang Y.-H. and Segur H., *Physica D: Nonlinear Phenom.*, **51** (1991) 109.
[9] Gaeta G., *J. Phys. A: Math. Gen.*, **27** (1994) 437.
[10] Gaeta G., *Phys. Rev. A*, **72** (2005) 033419.
[11] Gaeta G., Levi D. and Mancinelli R., *J. Nonlinear Math. Phys. Suppl.* 2, **12** (2005) 137.
[12] Gaeta G. and Mancinelli R., *Int. J. Geom. Methods Mod. Phys.*, **2** (2005) 1081.
[13] Gaeta G. and Mancinelli R., *J. Nonlinear Math. Phys.*, **12** (2005) 550.
[14] Goldenfeld N., Martin O., Oono Y. and Liu F., *Phys. Rev. Lett.*, **64** (1990) 1361.
[15] Iwasa M. and Nozaki K., *Prog. Theor. Phys.*, **116** (2006) 605.
[16] McCarthy P. J., *Phys. Lett. A*, **174** (1993) 25; McCarthy P.J., *Proc. R. Soc. London, Ser. A, Math. Phys. Eng. Sci.*, **358** (1978) 495.
[17] Penrose R., *Phys. Rev. Lett.*, **10** (1963) 66.
[18] Sachs R., *Phys. Rev.*, **128** (1962) 2851.
[19] Levi D. and Winternitz P., *J. Phys. A: Math. Gen.*, **39** (2006) R1.
[20] Baldwin D. and Hereman W., *J. Nonlinear Math. Phys.*, **13** (2006) 1.