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A nonstationary form of the range refraction parabolic equation and its application as an artificial boundary condition for the wave equation in a waveguide

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Abstract – The time-dependent form of Tappert's range refraction parabolic equation is derived and proposed as an artificial boundary condition for the wave equation in a waveguide. The numerical comparison with Higdon's absorbing boundary conditions shows sufficiently good quality of the new boundary condition at low computational cost.

Introduction. – We consider wave propagation in a two-dimensional waveguide $\Omega = \{(x,y)|\, a<y<b\}$ described by the wave equation

$$L\,u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \qquad (1)$$

where the sound speed c is a function of spatial variables, $c=c(x,y)$. For various purposes one needs to obtain one way or unidirectional wave equations, which permit wave propagation only in certain directions along the x-axis. For example, for such equations we can consider x as the evolution variable and set the initial boundary value problems with initial data at $x=x_0$, which are not well posed for the wave equation itself.

The main approach to this problem consists in factorization of the wave operator in eq. (1) into unidirectional pseudodifferential factors

$$L = L^+\cdot L^-, \quad L^\pm = \frac{\partial}{\partial x} \pm D, \quad D = \sqrt{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)}, \qquad (2)$$

where the positive square root is used. Then various approximations to the square root are used for obtaining concrete unidirectional equations. Note that the factorization in eq. (2) is not exact when c depends on x, so this approach requires optional assumptions of the asymptotic character on this dependence. In fact,

$$L = L^+\cdot L^- + [D,\,\partial/\partial x] = L^+\cdot L^- - D_x,$$

so it is sufficient to assume that c depends on x slowly, $c=c(\delta x, y)$ for some small parameter δ. Then $D_x = O(\delta)$. In this paper we also assume that c is regular enough with respect to y to perform the calculations below.

In the approximation of the square root in eq. (2) the assumption of small angle propagation with respect to the x-axis is usually used, *i.e.* the operator $(1/c^2)(\partial^2/\partial t^2) = A$ is treated as $O(1)$ and $(\partial^2/\partial y^2) = \epsilon B$ is $O(\epsilon)$ in some small parameter ϵ. In the case of constant sound speed c, when the operators A and B commute, the application of usual rational approximations to the square root function then yields local (differential) unidirectional wave equations. Under the assumption that $X = A + \partial^2/\partial y^2 - K$ is small for some constant K, the usual rational approximations with respect to X can also be applied to approximate the square root operator and its exponential (one-step propagator). This approach is well developed for the time-harmonic case [1–3].

With the strong dependence of c on y the operators A and B do not commute and the applicability of the standard rational approximations to the square root operator in eq. (2) is unclear. In fact, difficulties arise already in consideration of the first-order (linear in B) approximation. As we know now, this problem in a particular case was first solved correctly by R. Feynman, who in his

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paper [4] derived the formula

$$e^{A+\epsilon B} = e^{A} + \epsilon \int_0^1 e^{(1-s)A} B e^{sA}\, ds + O(\epsilon^2). \quad (3)$$

In the late 1970s, Tappert in his paper [5] derived the so-called range refraction parabolic equation for the time-harmonic sound propagation in waveguides with arbitrary dependence of the index of refraction on depth:

$$i v_x + \frac{1}{2k_0}\frac{\partial}{\partial y}\left(\frac{1}{n}\frac{\partial v}{\partial y}\right) + k_0\left[n + \frac{1}{4k_0^2}\left(\frac{n_y^2}{n^3} - \frac{n_{yy}}{n^2}\right)\right] v = 0, \quad (4)$$

where n is the index of refraction and k_0 is the reference wave number. He used the expansion of the operator square root in the form very similar to eq. (3) (see eq. (7) below), but without any derivation. As Feynman's paper was also not cited, the derivation of eq. (4) in Tappert's paper seemed to be somewhat mysterious.

In the next section we derive the nonstationary form of the Tappert's equation, using the same expansion, but derived from some elementary results of noncommutative analysis [6]. We hope that this information will be useful for the reader.

Note that Tappert's equation (4) is of restricted use for the computation of the wave field because it is not an *amplitude* equation. The same is true for our new eq. (10). As an application of the new equation we consider its use as an artificial (absorbing, nonreflecting) boundary condition (see, *e.g.*, [7–11]) at the boundaries $x = x_0$. These artificial boundaries and the corresponding boundary conditions are needed for restricting the computational domain to simulate waves propagating freely in the whole waveguide. Such a use is possible because any unidirectional equation partially annihilates waves propagating in the opposite direction.

Note that an analogous problem where the variability of coefficients is also essential was studied by a different method in [12,13].

Derivation of the time-dependent form of Tappert's range refraction parabolic equation. – Due to the equality

$$\int_0^1 f'(sx + (1-s)y)\, ds = \frac{f(x) - f(y)}{x - y}$$

the first-order term in eq. (3), omitting ϵ, is

$$\int_0^1 e^{(1-s)A} B e^{sA}\, ds = \overset{2}{B} \int_0^1 e^{(1-s)\overset{1}{A}} e^{s\overset{3}{A}}\, ds = \overset{2}{B}\, \frac{e^{\overset{3}{A}} - e^{\overset{1}{A}}}{\overset{3}{A} - \overset{1}{A}}$$

where the numbers above the operators (now called the Feynman numbers) define the order in which they are to operate [4,6]. "Thus, BA may be written $\overset{2}{B}\overset{1}{A}$ or $\overset{1}{A}\overset{2}{B}$" [4].

The same formula holds for some general class of operator functions f

$$f(A + \epsilon B) = f(A) + \epsilon\, \overset{2}{B}\, \frac{f(\overset{1}{A}) - f(\overset{3}{A})}{\overset{1}{A} - \overset{3}{A}} + O(\epsilon^2), \quad (5)$$

as was proved few years later by Daletskiy and Krein (see [6]).

Using eq. (5), the operator C in the expansion

$$(A + \epsilon B)^{1/2} = A^{1/2} + \epsilon C + O(\epsilon^2) \quad (6)$$

can be directly computed:

$$C = \overset{2}{B}\, \frac{\overset{1}{A}{}^{1/2} - \overset{3}{A}{}^{1/2}}{\overset{1}{A} - \overset{3}{A}} = \overset{2}{B}\, \frac{1}{\overset{1}{A}{}^{1/2} + \overset{3}{A}{}^{1/2}} =$$
$$\overset{2}{B} \int_0^\infty e^{-s\overset{1}{A}{}^{1/2}} e^{-s\overset{3}{A}{}^{1/2}}\, ds = \int_0^\infty e^{-sA^{1/2}} B e^{-sA^{1/2}}\, ds. \quad (7)$$

This formula can be verified by squaring both sides of eq. (7), which gives the equation for C

$$A^{1/2} C + C A^{1/2} = B.$$

Substitution into the last formula the expression for C from eq. (7) immediately gives

$$A^{1/2} C + C A^{1/2} = -\int_0^\infty \frac{d}{ds}\left(e^{-sA^{1/2}} B e^{-sA^{1/2}}\right) ds = B.$$

The last equality and the representation (7) holds because

$$e^{-sA^{1/2}} u = \exp\left(-s\frac{1}{c}\frac{\partial}{\partial t}\right) u = u(t - \frac{1}{c}s, x, y), \quad (8)$$

and the functions under consideration vanish with their derivatives before some time moment, say, for $t < 0$.

By a straightforward calculation using eq. (8) we first obtain

$$\begin{aligned} Cu = & \int_0^\infty \frac{(c_y)^2}{c^4} \cdot s^2 \cdot u_{tt}\left(t - 2\frac{1}{c}s, x, y\right) ds \\ & + \int_0^\infty \left(\frac{c_{yy}}{c^2} - 2\frac{(c_y)^2}{c^3}\right) \cdot s \cdot u_t\left(t - 2\frac{1}{c}s, x, y\right) ds \\ & + \int_0^\infty u_{yy}\left(t - 2\frac{1}{c}s, x, y\right) ds \\ & + 2\int_0^\infty u_{yt}\left(t - 2\frac{1}{c}s, x, y\right) \cdot s\frac{c_y}{c^2}\, ds. \end{aligned} \quad (9)$$

Then, using the formula

$$u_t\left(t - 2\frac{1}{c}s, x, y\right) = -\frac{c}{2}\frac{\partial}{\partial s} u\left(t - 2\frac{1}{c}s, x, y\right)$$

and the analogous formula for u_{tt}, we eliminate the time derivatives and s in eq. (9) by integration by parts as follows:

$$\int_0^\infty s^2 \cdot u_{tt}\left(t-2\frac{1}{c}s,x,y\right)\,\mathrm{d}s = c\int_0^\infty s\cdot u_t\left(t-2\frac{1}{c}s,x,y\right)\,\mathrm{d}s = \frac{c^2}{2}\int_0^\infty u\left(t-2\frac{1}{c}s,x,y\right)\,\mathrm{d}s.$$

At last, changing the variable s by $s_{new}=2s_{old}/c$, we find the final expression for the operator C and so the time-dependent form of Tappert's equation for right- and left-propagating waves

$$u_x \pm \frac{1}{c}u_t \mp \frac{1}{4}\left(c_{yy}-\frac{c_y^2}{c}\right)\int_0^\infty u(t-s,x,y)\mathrm{d}s \mp\frac{1}{2}\int_0^\infty \left(cu_y(t-s,x,y)\right)_y\,\mathrm{d}s = 0. \quad (10)$$

Differentiating eq. (10) with respect to t using the above technique, we obtain its differential form

$$u_{xt} \pm \frac{1}{c}u_{tt} \mp \frac{1}{4}\left(c_{yy}-\frac{c_y^2}{c}\right)u \mp \frac{1}{2}\left(cu_y\right)_y = 0, \quad (11)$$

which is, of course, not completely equivalent to eq. (10), especially in computational aspects. Equation (10) is more easily handled and, due to its nonlocality, has better stability properties than eq. (11).

It is easy to check that for the time harmonic $u=\mathrm{e}^{\mathrm{i}\omega t}v(x,y)$ eq. (11) coincides with the original range refraction parabolic equation for the field v (4). Conversely, the Fourier transform with respect to time of eq. (4), multiplied by $\mathrm{e}^{\mathrm{i}\omega t}$, gives eq. (11) and then, after integration with respect to t, eq. (10).

Equation (10) as an artificial boundary condition. – In this paper we will apply eq. (10) for left-going waves as an artificial boundary condition on the artificial boundary $x=x_0$, the computational domain is located in $x>x_0$. As was mentioned in the introduction, the problem consists in simulating waves, propagated in the unbounded waveguide, by solving some initial boundary value problem in the restricted computational domain.

As far as we know, only the technique of perfectly matched absorbing layers can be immediately applied to the problem under consideration in the case of waveguides with variable sound speed c (see [8]). Besides this technique the most frequently used by the practitioners boundary conditions have the form

$$\left[\prod_{j=1}^{J}\left(\frac{\partial}{\partial t}+C_j\frac{\partial}{\partial x}\right)\right]u=0, \quad (12)$$

where C_j are some constants. In the paper [14] Higdon proved (in the case of constant c) that the class of these conditions contains all boundary conditions obtained by the Padé approximations of the operators $L^\pm$ (the Engquist-Majda boundary conditions [7]). It is immediately seen that the condition (12) perfectly annihilates waves of the form $f(Ct-x)\phi(y)$ if the phase speed C is equal to some C_j. The boundary conditions (12) are referred in the literature as the Higdon conditions of order J.

The main problem in any practical use of the Higdon conditions consists in finding some (quasi)optimal set $\{C_j\}$. For waves with given frequency spectrum it is possible to try the phase or group velocities of the corresponding normal modes. The automatic choice, described in [15], also gives good results for dispersive waves. In essentially nonstationary problems for the wave equation the correct choice of C_j is difficult and the optimal set $\{C_j\}$ may depend on time. So in such ambiguous situations the use of some mean value of the sound speed c as C_j is probably more preferable.

As our boundary condition reduces to the second-order Engquist-Majda condition in the constant sound speed case, we expect that it will work better than the Higdon condition of order 2 in the variable sound speed case. The performed numerical experiments confirm this conclusion.

We must make some remarks on the well-posedness of the introduced boundary condition. In the case of constant sound speed one can recognize in eq. (11) the second order Engquist-Majda boundary condition [7], which is proved to lead to a well-posed mixed initial boundary value problem for the wave equation. The proof consists in checking the *uniform Kreiss condition* (UKC) and is valid as well in the variable coefficient case (see, *e.g.*, [16]). As the third term in eq. (11) is unessential for the UKC, we may conclude that the boundary condition (11) leads to a well-posed mixed initial boundary value problem for the wave equation. As eq. (10) implies eq. (11), the same is true for the boundary condition (10).

Finite-difference discretization. – In our numerical experiments we have used the standard second-order explicit finite difference scheme for the wave equation on the uniform grid $\{t^i,x_j,y_l\}$

$$\frac{1}{c_{j,l}^2}D_t^+D_t^-u_{j,l}^i = D_x^+D_x^-u_{j,l}^i + D_y^+D_y^-u_{j,l}^i, \quad (13)$$

where $u_{j,l}^i=u(t^i,x_j,y_l)(t^i=i\tau, x_j=jh-h/2, y_l=lh, i=0,1\dots,j=1,2\dots,l=1,2\dots)$, $D_x^+u_{j,l}^i=(u_{j+1,l}^i-u_{j,l}^i)/h$, $D_x^-u_{j,l}^i=(u_{j,l}^i-u_{j-1,l}^i)/h$, and D_t^+,D_t^-,D_y^+,D_y^- are defined analogously.

We assume that u and all its derivatives vanish for all $t<0$. Under this assumption the boundary condition of the Tappert type (10) on the left boundary at $x=0$ are

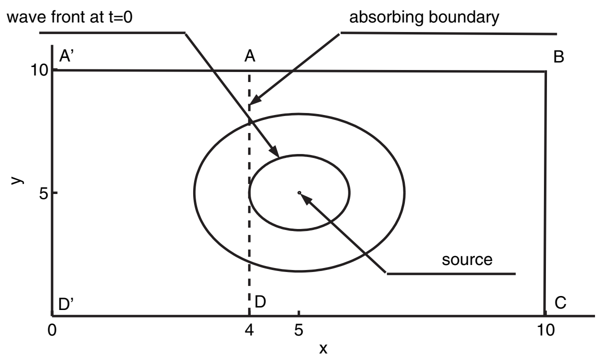

Fig. 1: Computational domain for the point source.

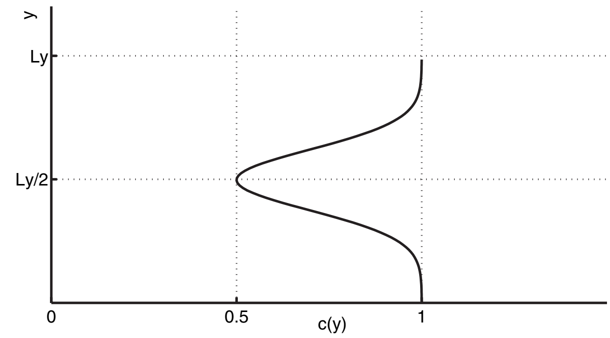

Fig. 2: $c(y)$ given by (18).

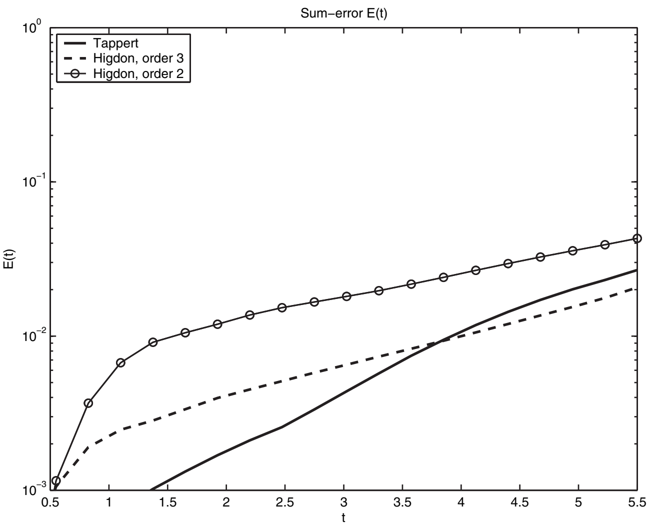

Fig. 3: Errors $E(t)$ for the first example.

discretized as

$$\begin{aligned}&D_x^+(u_{1,j}^i+u_{1,j}^{i+1})/2-D_t^+(u_{1,j}^i+u_{2,j}^i)/(2c_{3/2,j})\\&+\frac{\tau}{8}\left(c_{yy}-\frac{c_y^2}{c}\right)_{3/2,j}\cdot\sum_{m=2}^{i}(u_{1,j}^m+u_{2,j}^{m+1})\\&+\frac{\tau}{8}(c_y)_{3/2,j}\sum_{m=2}^{i}D_y^+D_y^-(u_{1,j}^m+u_{2,j}^{m+1})\\&+\frac{\tau c_{3/2,j}}{4}\sum_{m=2}^{i}D_y^+D_y^-(u_{1,j}^m+u_{2,j}^{m+1}),\end{aligned}\qquad(14)$$

where $(\cdot)_{3/2,j}$ is the value of $(\cdot)$ at $x=0, y=jl$. It should be noted that in practice the sound speed c is often given by some interpolation formula, so there is no need to calculate its derivatives by the difference methods.

The Higdon boundary conditions were discretized as described in [17].

The implementation of the boundary condition (10) in the finite element framework can also be easily obtained.

Numerical experiments. – The nondimensional computational domainfor these experiments is shown in fig. 1. It is a rectangle $\{x,y|\,0<x<l_x=10\,;\,0<y<L_y=10\}$.

At each boundary of the extended domain A′BCD′ the hard-wall (zero Neumann) boundary conditions are applied but waves do not reach these boundaries during calculations. At the boundary AD of the truncated domain ABCD the absorbing boundary conditions are imposed. Thus solving the initial boundary value problem for the wave equation first on the truncated domain and then on the extended one we can estimate the quality of solutions using error measures

$$E(t_i)=\frac{\sum\limits_j\sum\limits_k|u_{j,k}^i-u_E(t_i,x_j,y_k)|}{\sum\limits_j\sum\limits_k|u_E(t_i,x_j,y_k)|}\qquad(15)$$

and

$$e(t_i)=\max_j\max_k|u_{j,k}^i-u_E(t_i,x_j,y_k)|,\qquad(16)$$

where u is the solution of initial boundary value problem on the truncated domain, and u_E is the solution from the extended domain. As the initial conditions was taken the wave field in the homogeneous media produced by the point source of short duration located at the center of the extended domain.

Numerical solutions were obtained on the grid with the space step $\sigma=0.1$ and the sufficiently small time step τ which guarantees stability of the finite-difference scheme.

In our experiments we compare the new boundary condition, in the sequel called the Tappert condition, with the Higdon boundary conditions of order $J=2$ and $J=3$, with the $\{C_j\}$ given by

$$C_j=\frac{\int_0^{L_y}c(y)\,\mathrm{d}y}{L_y}.$$

For the first example we use $c(y)$ with a minimum at the depth $L_y/2$ given by (fig. 2)

$$c(x)=1-0.5e^{(y-L_y/2)^2/3},\qquad(17)$$

which gives $C_j\approx0.8565$. The point source duration was taken to be 1.4. The results of calculations are presented in fig. 3 and fig. 4.

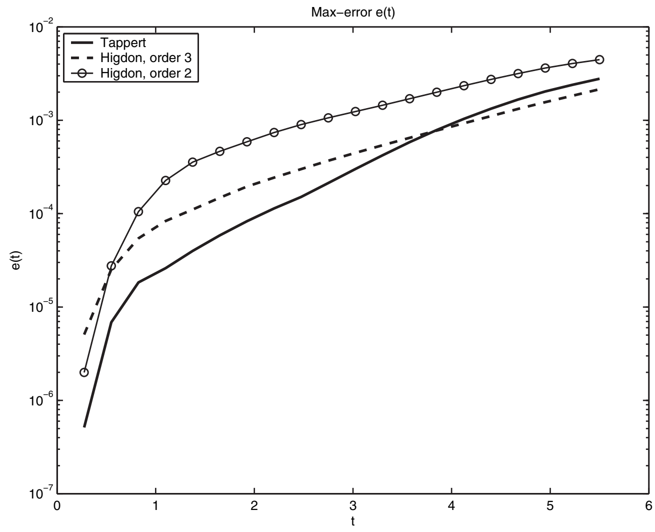

Fig. 4: Errors $e(t)$ for the first example.

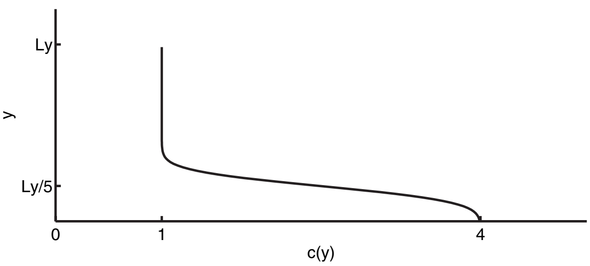

Fig. 5: $c(y)$ given by (18).

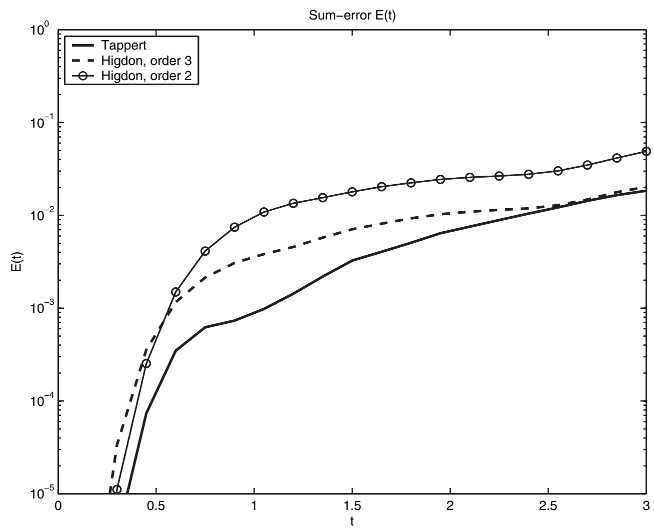

Fig. 6: Errors $E(t)$ for the second example.

In the second example the variation of $c(y)$ is larger (fig. 5), where $c(y)$ is defined through

$$c(y) = 4 - \frac{3}{\sqrt{\pi}} \int_{-\infty}^{y} e^{(s-L_y/5)^2}\, \mathrm{d}s. \tag{18}$$

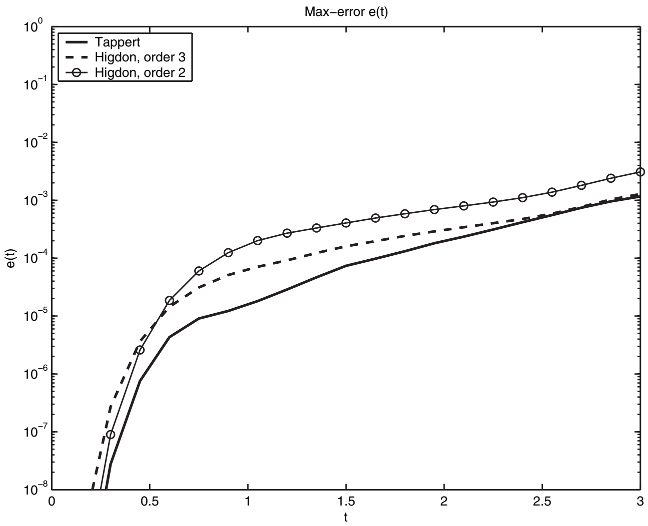

Fig. 7: Errors $e(t)$ for the second example.

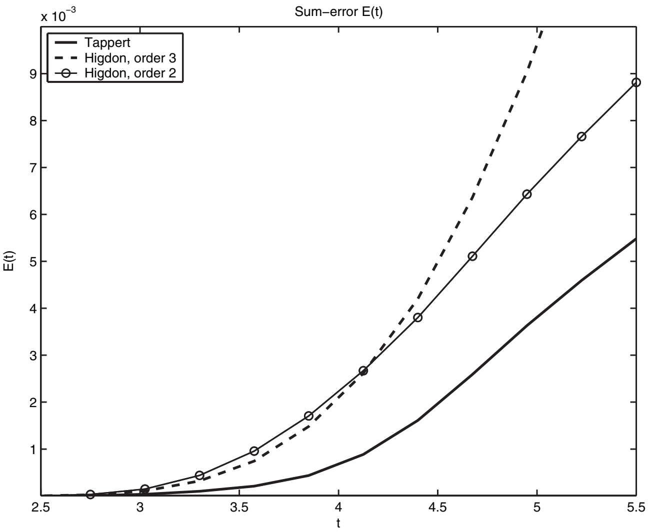

Fig. 8: Errors $E(t)$ for the third example.

In this example $C_j \approx 1.6521$ and the point source duration was taken to be 1.

The corresponding results are presented in fig. 6 and fig. 7.

We see that the larger the variation of the sound speed is, the better the Tappert condition works.

As the Tappert condition (10) contains no derivatives of the sound speed with respect to the range variable, it can be applied in the range-dependent environment without any modifications. In the third example the following range-dependent sound speed $c(x,y)$ was taken:

$$c(x, y) = 1 - 0.5e^{(x-L_x/2)^2/3}. \tag{19}$$

In this example the source was positioned at $x = 3L_x/4$, $y = L_y/2$. The results are presented in fig. 8 and fig. 9. This and various other experiments show that in the range-dependent case the growth of errors

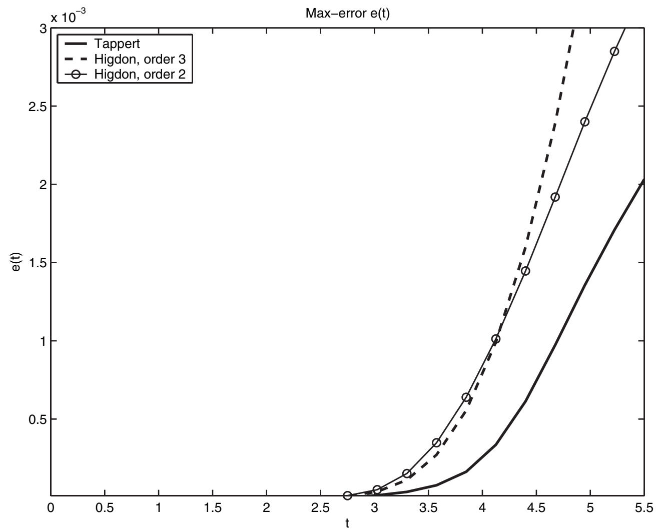

Fig. 9: Errors $e(t)$ for the third example.

under the Tappert condition (10) is essentially the same as in the range-independent examples.

The discrete form of the Tappert boundary condition (14) shows that the integrals in eq. (10) are simply accumulated at each time step, so the computational cost of the new boundary condition only slightly differs from those of the first-order Higdon condition. The computational cost of the second-order and the third-order Higdon conditions is about 15% and 40% greater. These estimates are very approximate because the calculations were conducted in the MATLAB framework.

At last we should note that the behavior of the new boundary condition over long times is essentially as good (or bad) as those of the Higdon conditions.

Conclusion. – In this paper the time-dependent form of Tappert's range refraction parabolic eq. (10) is derived. This equation describes unidirectional, small angle (with respect to the x-axis) wave propagation and can be used for solving various problems of nonstationary-wave propagation which are, in particular, ill-posed for the wave equation (1). Despite its nonlocality in time, it is easily calculated.

Here the new equation was applied as an artificial boundary condition for the wave equation in a stratified waveguide. Numerical experiments, performed for comparison of the new boundary condition with Higdon's absorbing boundary conditions, showed its sufficiently good quality at low computational cost.

Equation (10), regarded as an artificial boundary condition, can be considered as the first member of a new sequence of boundary conditions which are suitable for stratified waveguides. This sequence can be obtained by the calculation of the following terms in expansion of the square root operator into the Newton series (noncommutative analog of the Tailor series, see [6]). To achieve good stability properties the regularization of obtained in such a way partial sums will be necessary. This can be done by the application of some form of noncommutative rational approximation (*e.g.*, Padé appoximation ([18], Part 2, sect. 1.5)) and will lead also to the *wide angle* extensions of eq. (10).

The problems for further investigations consist in the calculation of the following members of this sequence, verification their well-posedness and automatization of their numerical implementation. We intend to consider some of these problems in our future studies. The results of our future studies will be compared with the results of the recent works [19,20] on the Higdon boundary conditions, where the problem of variable coefficients was also considered.

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