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Unified description of pairing, trionic and quarteting states for one-dimensional SU(4) attractive fermions

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Abstract – Paired states, trions and quarteting states in one-dimensional $SU(4)$ attractive fermions are investigated via exact Bethe ansatz calculations. In particular, quantum phase transitions are identified and calculated from the quarteting phase into normal Fermi liquid, trionic states and spin-2 paired states which belong to the universality class of linear field-dependent magnetization in the vicinity of critical points. Moreover, unified exact results for the ground-state energy, chemical potentials and complete phase diagrams for isospin $S=1/2$, 1, 3/2 attractive fermions with external fields are presented. Also identified are the magnetization plateaux of $m^z=M_s/3$ and $m^z=2M_s/3$, where M_s is the magnetization saturation value. The universality of finite-size corrections and collective dispersion relations provides a further test ground for low-energy effective field theory.

Experimental advances with higher-spin fermionic systems of ultracold atoms present a unique opportunity to rigorously test the current understanding of molecular superfluids and more generally to probe the nature of quantum many-body systems. Three-component ultracold fermions give rise to a phase transition from three-body bound states called trions into the BCS pairing state [1–4]. In particular, spin-(3/2) interacting atomic fermions have been predicted to exhibit a quarteting phase, *i.e.*, a bound state of two BCS pairs [2,4,5], and BCS pairing with total spin 2 [6]. Such molecular superfluids have currently received considerable interest in the context of the one-dimensional (1D) lattice Hubbard model of ultracold atoms [2,4,7,8] and degenerate quantum gases [9–12] due to new progress in experimental realization of highly degenerate atomic gases [13].

Furthermore, recent experiments on systems of ultracold atoms confined to one dimension (1D) [14–16] have revived interest in Bethe ansatz (BA) integrable models of interacting bosons and multi-component fermions. The nature of "no diffraction" in the many-body scattering matrix of 1D integrable models results in key features of quantum many-body physics which are specified by the phenomena of spin-charge separation, scaling dimensions and universality classes of quantum phase transitions and criticality [17–19]. In this letter, we study complete phase diagrams and quantum phase transitions in integrable 1D $SU(4)$ attractive fermions with external magnetic fields. We demonstrate that quantum phase transitions from quarteting states into phases of normal Fermi liquid, trions and spin paired states are fully controlled by Zeeman splittings. In particular, the unified exact results obtained for isospin $S=1/2$, 1, 3/2 attractive fermions display a simplicity and universality which gives insight in understanding high-spin paired states and spin liquid behaviour in multi-component interacting fermions.

The model. – We consider a δ-function (contact potential) interacting system of N atomic fermions with equal mass m which may occupy four possible

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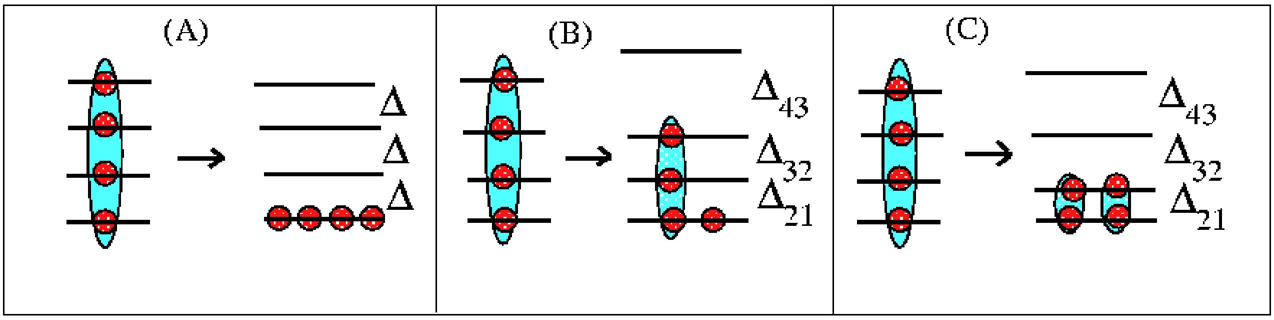

Fig. 1: (Colour on-line) Depiction of phase transitions from quarteting states into (A) normal Fermi liquid, (B) trions and (C) bound pairs. Ellipses denote spin-neutral bound states.

hyperfine levels ($|i\rangle$, $i=1,\ldots,4$) labeled by isospin states $\pm 3/2$, $\pm 1/2$ and constrained by periodic boundary conditions to a line of length L. The Hamiltonian [20,21] reads $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I+E_z$ with kinetic energy $\mathcal{H}_0=-\frac{\hbar^2}{2m}\times\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}$, interaction energy $\mathcal{H}_I=g_{1\mathrm{D}}\sum_{1\leqslant i<j\leqslant N}\delta(x_i-x_j)$ and Zeeman energy $E_z=\sum_{i=1}^{4}N^i\epsilon_Z^i(\mu_B^i,B)$. Here N^i is the number of fermions in state $|i\rangle$ with Zeeman energy ϵ_Z^i (acting as a chemical potential μ_i [3]) determined by the magnetic moments μ_B^i and the magnetic field B. The spin-independent contact interaction $g_{1\mathrm{D}}$ remains between fermions with different hyperfine states and preserves the spins in each hyperfine states, *i.e.*, N^i with $i=1,\ldots,4$ are good quantum numbers. Although these conditions appear somewhat restrictive, utilizing the broad Feshbach resonances, it is possible to tune scattering lengths between different states close to each other to form high-degeneracy Fermi gases [13,14,22]. Thus the model still captures the essential physics relevant to the multiple phases of molecular superfluids. The coupling constant $g_{1\mathrm{D}}=-\hbar^2c/m$ with interaction strength $c=-2/a_{1\mathrm{D}}$ determined by the effective 1D scattering length $a_{1\mathrm{D}}$ [23]. For simplicity, we choose the dimensionless units of $\hbar=2m=1$ and use the dimensionless coupling constant $\gamma=c/n$ with linear density $n=N/L$.

For an irreducible representation $[4^{N_4}3^{N_3}2^{N_2}1^{N_1}]$, a four-column Young tableau encodes the numbers of unpaired fermions (N_1), bound pairs (N_2), trions (N_3) and quarteting states (N_4) with $N_i=N^i-N^{i+1}$ and $N^5=0$. For convenience in calculation, we rewrite the Zeeman energy as $E_Z=-\sum_{i=0}^{3}H_iN_i$. Here $H_0=\sum_{i=1}^{4}\epsilon_z^i/4$ is irrelevant because $N_0=N$ is fixed. The other values H_1,H_2,H_3 are related to Zeeman splittings $\Delta_{i+1\,i}=\epsilon_Z^{i+1}-\epsilon_Z^i$ with $i=1,2,3$ via the relation

$$\begin{pmatrix}H_1\\H_2\\H_3\end{pmatrix}=\frac{1}{4}\begin{pmatrix}3&2&1\\2&4&2\\1&2&3\end{pmatrix}\begin{pmatrix}\Delta_{21}\\\Delta_{32}\\\Delta_{43}\end{pmatrix}. \qquad (1)$$

We shall find that equally spaced (linear) Zeeman splitting, *i.e.*, $\Delta_{a+1\,a}=\Delta$ for $a=1,2,3$, drive a smooth phase transition from a quarteting phase into a normal Fermi liquid, as depicted in part (A) in fig. 1. Unequally spaced (nonlinear) Zeeman splittings may trigger spin-neutral bound states which are illustrated in parts (B) and (C) in fig. 1. It follows from the relation (1) that for linear Zeeman splitting, the magnetic moments of a trion, a bound pair and the unpaired fermion are $\frac{3}{2}$, 2 and $\frac{3}{2}$, respectively. The quarteting state remains a spin singlet.

The energy eigenspectrum is given in terms of the quasimomenta k_j of the fermions via $E=\sum_{j=1}^{N}k_j^2$, which satisfy the BA equations [20,21]

$$\begin{aligned}&\exp(\mathrm{i}k_jL)=\prod_{l=1}^{M_1}\frac{k_j-\Lambda_l^{(1)}+\mathrm{i}\frac{1}{2}c}{k_j-\Lambda_l^{(1)}-\mathrm{i}\frac{1}{2}c}\\&\prod_{\beta=1}^{M_{\ell-1}}\frac{\Lambda_\alpha^{(\ell)}-\Lambda_\beta^{(\ell-1)}+\mathrm{i}\frac{1}{2}c}{\Lambda_\alpha^{(\ell)}-\Lambda_\beta^{(\ell-1)}-\mathrm{i}\frac{1}{2}c}=\\&-\prod_{\gamma=1}^{M_\ell}\frac{\Lambda_\alpha^{(\ell)}-\Lambda_\gamma^{(\ell)}+\mathrm{i}c}{\Lambda_\alpha^{(\ell)}-\Lambda_\gamma^{(\ell)}-\mathrm{i}c}\prod_{\delta=1}^{M_{\ell+1}}\frac{\Lambda_\alpha^{(\ell)}-\Lambda_\delta^{(\ell+1)}-\mathrm{i}\frac{1}{2}c}{\Lambda_\alpha^{(\ell)}-\Lambda_\delta^{(\ell+1)}+\mathrm{i}\frac{1}{2}c}. \qquad (2)\end{aligned}$$

Here $j=1,\ldots,N$ and $\alpha=1,\ldots,M_\ell$. The parameters $\Lambda_\alpha^{(\ell)}$ with $\ell=1,2,3$ are the spin rapidities, where we denote $\Lambda_\alpha^{(0)}=k_\alpha$ and $\Lambda_\alpha^{(5)}=0$. The quantum numbers are given by $M_i=\sum_{j=i}^{3}N_{i+1}$, $M_0=N$.

Charge bound states. – For attractive interaction, the BA equations (2) admit charge bound states and spin strings. In particular, the $SU(4)$ symmetry acquires three kinds of charge bound states: quarteting states, trions and bound pairs. The patterns of these bound states and spin strings determined by (2) underpin the nature of quantum statistics and many-body effects in the atomic system.

In the weak-coupling regime, *i.e.*, $L|c|\ll 1$, we find that the imaginary parts $\mathrm{i}y$ of the charge bound states are the roots of Hermite polynomials H_k of degree k. Specifically, $H_k(\frac{L}{2|c|}y)=0$, with $k=2,3,4$ for a bound pair, a trion and a quarteting state, respectively. This result is indicative of a universal signature of many-body cooperative effects driven by dynamical interaction. The real parts of the quasimomenta deviate slightly from the values determined by Fermi statistics for the $c=0$ case. In this weak-coupling limit, the BA equations (2) reduce to Gaudin model-like BA equations [24] which describe the multiple-charge bound-state scattering. Using these root patterns, we explicitly obtain the energy (in units of $\hbar^2/2m$)

$$\frac{E}{L} \approx \frac{\pi^2}{3}\sum_{k=1}^{\kappa} k n_k^3 + \pi^2 \sum_{j=1}^{\kappa-1} j n_j \sum_{k=j+1}^{\kappa} n_k \sum_{\ell=j}^{\kappa} n_\ell \\ -|c|\left(2\sum_{i=1}^{\kappa-1}\sum_{j=i+1}^{\kappa} i(j-1)n_i n_j + \sum_{j=1}^{\kappa} j(j-1)n_j^2\right). \quad (3)$$

This result unifies the ground-state energy results for two-, three- and four-component fermions (where $\kappa=2,3,4$, respectively) for weakly attractive interaction. In this result the densities of unpaired fermions, BCS pairs, trions, and qarteting states are denoted by $n_a = N_a/L$ with $a=1,2,3,4$, respectively. The ground-state energy (3) is dominated by the kinetic energy of composite particles and unpaired fermions and has a mean-field theory configuration, where the interaction energy accounts for density density interaction between charge bound states and between charge bound states and unpaired fermions. For the weak-coupling limit, spin-neutral bound states are unstable against thermal and spin fluctuations. They form a gapless superconducting phase.

On the other hand, for the strongly attractive regime $L|c| \gg 1$, the imaginary parts of the bound states become equal-spaced, *i.e.*, a qarteting state has the form $k_i = \Lambda_i^{(3)} \pm \mathrm{i}3|c|/2$, $\Lambda^{(3)} \pm \mathrm{i}|c|/2$; the trion state is $k_j = \Lambda_j^{(2)} \pm \mathrm{i}|c|, \lambda_j^{(2)}$ and for the bound pair $k_r = \Lambda_r^{(1)} \pm \mathrm{i}|c|/2$. The corresponding binding energies are given by $\epsilon_\ell = c^2\ell(\ell^2-1)/12$ with $\ell = 1,2,3$, respectively. Substituting these bound-state root patterns into the BA equations (2), we explicitly obtain their real parts $\Lambda^{\ell}_{i,j,r}$, with $\ell=1,2,3$. This leads to a unified expresion

$$\frac{E}{L} \approx \sum_{k=1}^{4} \frac{\pi^2 n_k^3}{3k}\left(1+\frac{2}{|c|}A_k + \frac{3}{c^2}A_k^2\right) - \sum_{\ell=2}^{4} n_\ell \epsilon_\ell \quad (4)$$

for the ground-state energy for two-, three- and four-component attractive fermions (in units of $\hbar^2/2m$) where the functions $A_1 = 4n_2 + 2n_3 + \frac{4n_4}{3}$, $A_2 = 2n_1 + n_2 + \frac{8n_3}{3} + \frac{3n_4}{2}$, $A_3 = \frac{2n_1}{3} + \frac{16n_2}{9} + n_3 + \frac{92n_4}{45}$ and $A_4 = \frac{n_1}{3} + \frac{3n_2}{4} + \frac{23n_3}{15} + \frac{11n_4}{12}$. The functions A_k reveal the scattering signature in different channels, for example, n_1 does not appear in A_1 due to the lack of s-wave scattering for unpaired fermions. We note that for two-component attractive fermions the terms involving n_3 and n_4 should be excluded [25] whereas for three-component attractive fermions n_4 does not appear. The unified structure of the ground-state energy (4) can be amended with appropriate A_k functions for higher-spin fermions.

Charge bound states in equilibrium. – The BA equations (2) in principle give the complete quantum states of the model. However, at finite temperatures, the true physical states become degenerate. In the thermodynamic limit, $L, N \to \infty$ with N/L fixed, the grand partition function is given by $Z = \mathrm{tr}(\mathrm{e}^{-\mathcal{H}/T}) = \mathrm{e}^{-G/T}$ [17,26,27], where the Gibbs free energy $G = E + E_{\mathrm{Z}} - \mu N - TS$, the chemical potential μ, the Zeeman energy $E_{\mathrm{Z}} = -\sum_{i=0}^{3} H_i N_i$ and the entropy S are given in terms of densities of charge bound states and spin strings subject to the BA equations (2). The equilibrium states are determined by the minimization of the Gibbs free energy, which gives rise to a set of coupled nonlinear integral equations —the thermodynamic Bethe ansatz (TBA) equations. Following the TBA treatment for spin-(1/2) attractive fermions [17], we obtain the dressed energy equations

$$\begin{aligned}
\epsilon^{(4)}(\tau) &= 4(\tau^2-\mu^4) - a_3 * \epsilon^{(1)}(\tau) - [a_{2,4}] * \epsilon^{(2)}(\tau) \\
&\quad - [a_{1,3,5}] * \epsilon^{(3)}(\tau) - [a_{2,4,6}] * \epsilon^{(4)}(\tau), \\
\epsilon^{(3)}(\lambda) &= 3(\lambda^2-\mu^3) - a_2 * \epsilon^{(1)}(\lambda) - [a_{1,3}] * \epsilon^{(2)}(\lambda) \\
&\quad - [a_{2,4}] * \epsilon^{(3)}(\lambda) - [a_{1,3,5}] * \epsilon^{(4)}(\lambda), \\
\epsilon^{(2)}(\Lambda) &= 2(\Lambda^2-\mu^2) - a_1 * \epsilon^{(1)}(\Lambda) - a_2 * \epsilon^{2}(\Lambda) \\
&\quad - [a_{1,3}] * \epsilon^{(3)}(\Lambda) - [a_{2,4}] * \epsilon^{(4)}(\Lambda), \\
\epsilon^{(1)}(k) &= k^2 - \mu^1 - \textstyle\sum_{i=1}^{3} a_i * \epsilon^{(i+1)}(k),
\end{aligned} \quad (5)$$

which will be used to study quantum phase transitions at zero temperature. In these equations the function $a_j(x) = \frac{1}{2\pi}\frac{j|c|}{(jc/2)^2+x^2}$ and $a_j * \epsilon^{(a)}(x) = \int_{Q_a}^{Q_a} a_j(x-y)\epsilon^{(a)}(y)\mathrm{d}y$ is the convolution. Furthermore, we have used the abbreviation $a_{i,j,k} = a_i + a_j + a_k$. We denote the dressed energies $\epsilon^{(a)}$ with $a = 1,\ldots,4$ and the effective chemical potentials $\mu^a = H_a/a + \mu + c^2(a^2-1)/(12)$ with $H_4 = 0$ for unpaired fermions, pairs, trions and quarteting states, respectively. The negative part of the dressed energies $\epsilon^{(a)}(x)$ for $x \leqslant Q_a$ corresponds to the occupied states in the Fermi seas with the positive part of $\epsilon^{(a)}$ corresponding to the unoccupied states. The integration boundaries Q_m characterize the "Fermi surfaces" at $\epsilon^{(m)}(Q_m) = 0$.

The Gibbs free energy per unit length at zero temperature is given by $G = \sum_{m=1}^{4} \frac{m}{2\pi}\int_{-Q_m}^{Q_m} \epsilon^{(m)}(x)\mathrm{d}x$. The dressed energy energy equations (5) describe the band fillings with respect to Zeeman splittings H_i and chemical potentials and provide complete phase diagrams and quantum phase transitions for the model. Solving eqs. (5) by iteration among the relations $-\frac{\partial G}{\partial \mu} = n$, $-\frac{\partial G}{\partial H_i} = n_i$, $i=1,2,3$ and the Fermi point conditions $\epsilon^{(m)}(Q_m) = 0$ gives the effective chemical potentials

$$\frac{\mu^\kappa}{\pi^2} \approx \frac{n_\kappa^2}{k^2}\left(1 + \frac{2A_\kappa}{|c|} + \frac{3A_\kappa^2}{c^2}\right) + \frac{\vec{B}_\kappa \cdot \vec{I}}{|c|} + \frac{3\vec{B}_\kappa \cdot \vec{A}}{c^2}, \quad (6)$$

which characterize Fermi surfaces of stable spin-neutral states and unpaired states. In this equation we have introduced an inner product $\vec{B}_k \cdot \vec{A}$ with $\vec{A} = (A_1, A_2, A_3, A_4)$ and $\vec{B}_\kappa = (B_\kappa^1, B_\kappa^2, B_\kappa^3, B_\kappa^4)$. Here

$\vec{B}_4=(\frac{2n_1^3}{9},\frac{n_2^3}{8},\frac{46n_3^3}{405},\frac{11n_4^3}{288})$, $\vec{B}_3=(\frac{4n_1^3}{9},\frac{8n_2^3}{27},\frac{2n_3^3}{27},\frac{23n_4^3}{270})$, $\vec{B}_2=(\frac{4n_1^3}{3},\frac{n_2^3}{6},\frac{16n_3^3}{81},\frac{n_4^3}{16})$ and $\vec{B}_1=(0,\frac{2n_2^3}{3},\frac{4n_3^3}{27},\frac{n_4^3}{18})$. The A_i are as given above and $\vec{I}$ is the identity vector. The expression (6) for the chemical potentials μ^κ contains the previous results for isospin $S=1/2$, 1 Fermi gases [10,25,28]. We have found that the energy (4) derived from the discrete BA equations for arbitrary population imbalances can also be obtained from $E/L=\mu n+G+n_1H_1+n_2H_2+n_3H_3$. This indicates that the BA spin-neutral states comprise the equilibrium stable states in the thermodynamic limit.

Magnetism and quantum phase transitions. – The low-energy excitations split into collective excitations carrying charge and collective excitations carrying spin. This leads to the phenomenon of spin-charge separation. The charge excitations are described by sound modes with a linear dispersion. The spin excitations are gapped [7,9] with a dispersion relation $\epsilon_\nu(p)=\sqrt{\Delta_\nu^2+v_\nu^2p^2}$ where Δ_ν is the excitation gap and v_ν is the spin velocity in spin branch ν. However, for strong attractive interaction the low-energy physics is dominated by charge density fluctuations. This is because the spin wave fluctuations are fully suppressed by a large energy gap at low temperatures. This configuration is evidenced by the universality class of finite-size correction to the ground-state energy $E(L,N)-LE_0^\infty=-\frac{\pi\hbar C}{6L}\sum_{k=1}^4 v_k$, where the central charge $C=1$ for the $U(1)$ symmetry and charge velocities $v_k\approx\frac{\hbar\pi n_k}{mk}(1+\frac{2}{|c|}A_k)$ with $k=1,\ldots,4$ are the charge velocities for unpaired fermions and charge bound states. For the singlet ground state, the spin velocity $v_s\approx\frac{\sqrt{5}|c|}{\sqrt{2}}(1+\frac{1}{3|\gamma|})$ in the spin-(3/2) hyperfine branch, which is divergent due to the energy gap.

We find from the dressed energy equations (5) that quantum phase transitions driven by Zeeman splittings can be determined by three independent external field-energy transfer relations

$$\begin{aligned}H_1&=\frac{5}{4}c^2+u^1-u^4,\\ H_2&=2c^2+2(u^2-u^4),\\ H_3&=\frac{7}{4}c^2+3(u^3-u^4).\end{aligned}\tag{7}$$

The Fermi surfaces and charge bound states are fully controlled by the Zeeman splitting parameters. The complete phase diagrams are determined by eqs. (7) and certain combinations of these equations. Without loss of generality, we consider only terms up to order of $1/|\gamma|$ in the following analysis.

Using the energy transfer relations (7), we find that linear Zeeman splitting Δ lifts the $SU(4)$ symmetry to $U(1)^4$ symmetry for $|\gamma|\gg1$, *i.e.*, linear Zeeman splitting does not favor spin-neutral bound states (recall part (A) in fig. 1). The lower critical field $\Delta_{c1}\approx\frac{5c^2}{6}-\frac{n^2\pi^2}{384}(1+\frac{7}{18|\gamma|})$ diminishes the gap, thus the excitations becomes gapless. Using the definition of magnetization $m^z=M^z/M_s$ with $M^z=\frac{3}{2}n_1+2n_2+\frac{3}{2}n_3$ and $M_s=\frac{3}{2}n$, we see that in the vicinity of Δ_{c1}, the system exhibits a linear field-dependent magnetization of the form $m^z=192(\Delta-\Delta_{c1})/(n\pi^2)$ with a finite susceptibility $\chi=\partial m^z/\partial\Delta\approx192/(n\pi^2)$. This result provides a testing ground for low-energy field theory [18,19]. When the Zeeman splitting Δ is greater than the upper critical field $\Delta_{c2}\approx\frac{5c^2}{6}+\frac{2n^2\pi^2}{3}(1-\frac{2}{9|\gamma|})$ the system is fully polarized into a normal Fermi liquid. For the intermediate regime, $\Delta_{c1}<\Delta<\Delta_{c2}$, the quarteting state and unpaired fermions coexist. The phase transition at the critical point Δ_{c2} belongs the same linear field-dependent universality class. A plot of the magnetization *vs.* Zeeman splitting is given in fig. 2.

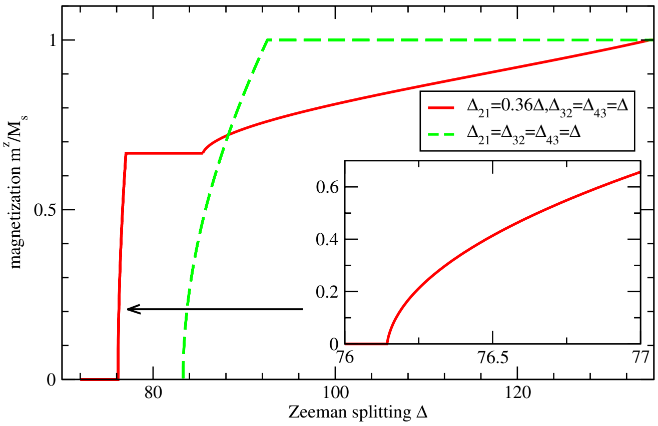

Fig. 2: (Colour on-line) Magnetization as a function of the Zeeman splitting parameter Δ. The curves shown, obtained from (7) with the effective chemical potentials given in (6), are for linear Zeeman splitting (dashed line) with $c=-10$ and $n=1$ and unequally spaced Zeeman splitting (solid line) with $c=-8$ and $n=1$. The 2/3 magnetization plateau occurs through subtle tuning of the Zeeman splitting parameters. Here, *e.g.*, $\Delta_{21}=f\Delta_{43}=f\Delta_{32}$ with $f=0.36$.

For nonlinear Zeeman splitting the quarteting state can break into two spin-2 bound pairs as depicted in part (C) of fig. 1. In order to trigger such a paired phase, we let $\Delta_{43}=\Delta_{32}$ in the relation (1). In fig. 3 we demonstrate the resulting interplay between the quantum phases of quarteting states (phase Q), spin-2 pairs (phase D) and unpaired fermions (phase U). We see clearly that the quarteting states are stable for $\Delta_{21}<\frac{5}{3}c^2-\frac{n^2\pi^2}{192}(1+\frac{7}{18|\gamma|})-\Delta_{43}$ and $\Delta_{43}<\frac{4}{3}c^2-\frac{n^2\pi^2}{192}(1+\frac{13}{36|\gamma|})-\frac{\Delta_{21}}{3}$. The phase diagram shown in fig. 3 is determined by the first two equations of (7) and $\Delta_{21}/2=c^2/4+\mu^1-\mu^2$ describing the mixed $D+U$ phase. In order to clearly see magnetization plateaux, we choose a simple linear relation $\Delta_{21}=f\Delta_{43}$. We then find that $f<\frac{3}{7}(1-\frac{5\pi^2}{14\gamma^2})$ and magnetization plateaux $m^z=0,2M_s/3,M_s$ occur, where $M_s=3n/2$ is the saturation magnetization (see fig. 2). This is obvious because the lines fixed by the slope f simultaneously pass the Q, D and U phases. Moreover, if the Zeeman splitting Δ_{32} is large enough, the phases D

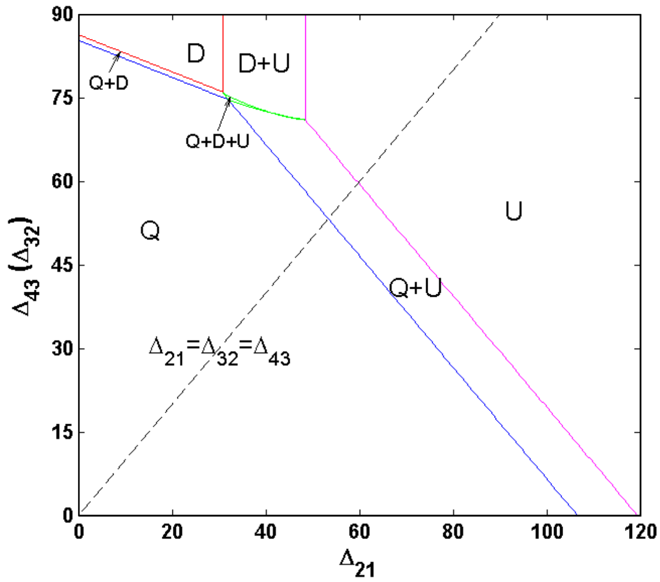

Fig. 3: (Colour on-line) Typical phase diagram in the Δ_{21}-$\Delta_{43}(\Delta_{32})$ plane with $\Delta_{43}=\Delta_{32}$ in the strong-coupling regime $c=-8$ and $n=1$. In this case we see clearly that Zeeman splitting can trigger a quantum phase transition from a quarteting state (phase Q) into two spin-2 bound pairs (phase D), as depicted schematically in part (C) of fig. 1. Other phases involved are unpaired fermions (U) and mixed phases.

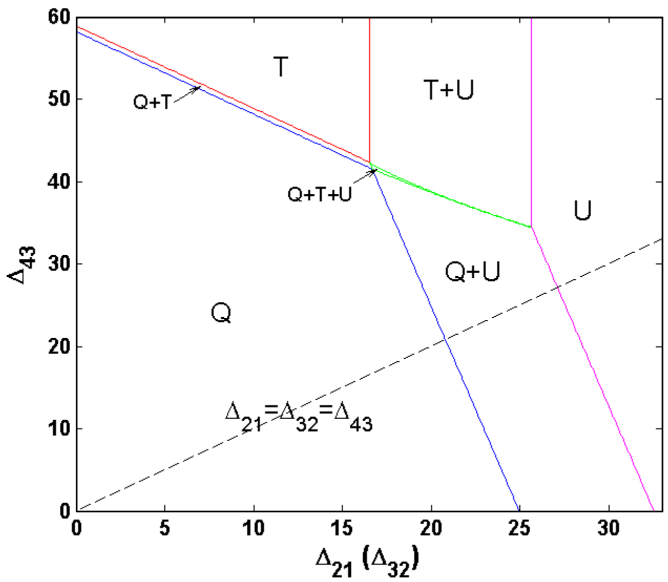

Fig. 4: (Colour on-line) Typical phase diagram in the $\Delta_{21}(\Delta_{32})$-Δ_{43} plane with $\Delta_{21}=\Delta_{32}$ in the strong-coupling regime $c=-5$ and $n=1$. In this case we see clearly that Zeeman splitting can trigger a quantum phase transition from quarteting states (phase Q) to trions (phase T), as depicted in part (B) of fig. 1. Other phases involved are unpaired fermions (U) and mixed phases.

and $D+U$ form the same phase diagram for spin-(1/2) interacting fermions with polarization [28].

If we set $\Delta_{21}=\Delta_{32}$ in the relation (1), the Zeeman parameters may trigger a phase transition from quarteting states into trions (recall part (B) in fig. 1). Figure 4 shows an exact phase diagram in the $\Delta_{21}(\Delta_{32})$-Δ_{43} plane. Varying the Zeeman splitting Δ_{43} and Δ_{21} reveals smooth phase transitions from quarteting states into trions (phase T) or a normal Fermi liquid (phase U). Similarly, we choose $\Delta_{43}=f\Delta_{21}$ and see that magnetization plateaux $m^z=0, M_s/3, M_s$ occur if $f>\frac{5}{2}(1+\frac{\pi^2}{18\gamma^2})$. The critical points for the plateaux can be analytically determined from the first and third equations in (7) and an additional equation $\Delta=\frac{2}{3}c^2+\mu^1-\mu^3$ for the mixed $T+U$ phase. Actually, if Δ_{43} is large enough, the model reduces to attractive fermions with three hyperfine levels [10]. We point out that all phase transitions are of second order with a linear field-dependent magnetization in the vicinities of critical points.

To conclude, we have presented unified exact results for complete phase diagrams and quantum phase transitions from quarteting states into spin-2 paired states and trions in 1D $SU(4)$ fermions with population imbalance. The ground-state properties and magnetic effects provide a testing ground for low-energy effective field theory and a bench mark for experiments with multicomponent ultracold fermionic atoms.

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