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Quantum amplitude amplification by phase noise

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Abstract – I consider a system in which an atom in a superposition of momentum eigenstates of a periodic potential is subject to periodic random phase jumps in that potential. If the j-th state in superposition is "marked" by giving the $(j-1)$-th state a phase flip, I show that for the correct amount of noise, a $\sim 6\,\mathrm{dB}$ amplification of the marked state probability amplitude occurs, with no significant amplification in other states.

Quantum state amplitude amplification is one of the cornerstones of quantum information processing [1]. The ability to selectively amplify states which hold information that we wish to know at the expense of ones that do not is an essential part of why quantum computers can give computational advantages over classical computers in spite of the limitations imposed by Born's rule (*i.e.* the probabilistic nature of quantum measurement [2]).

A particularly celebrated example of a quantum algorithm which employs quantum amplitude amplification in its operation is Grover's quantum search algorithm [3]. In this procedure, a quantum register is initialised in an equal amplitude superposition of all the qubits which are to be searched. The existence of a so-called "oracle" function is assumed which is able to identify the state being searched for, marking it by flipping its quantum phase relative to the other states. Then, an inversion-about-mean procedure is applied which *amplifies* the phase-flipped state at the expense of other states. Repeating this procedure $\sim \sqrt{N}$ times allows the marked state to be retrieved by a final measurement with close to 100% certainty. Although a large-scale version of the Grover algorithm has never been implemented, a few experiments have tested the phase flipping and inversion-about-mean procedures [4,5] for a number of qubits ~ 10. Aside from Grover's algorithm, applications of quantum amplitude amplification include quantum counting and amplitude estimation [6].

Moving now to the specific case of computing with neutral atoms, it may be seen that where a high degree of coherent control of quantum states is available, the ability to realise the Hamiltonian necessary to perform amplitude amplification may be lacking. Although numerous schemes exist for performing quantum computation in optical lattices [7] and considerable experimental progress has been made recently [8], performing operations such as the inversion-about-mean procedure requires delicate control over atomic interactions in optical lattices which is still very difficult to achieve. Along with these difficulties, the ever present problem of noise induced decoherence makes the realisation of sophisticated quantum amplification based algorithms a great challenge.

The purpose of the present paper is to introduce a surprising variation of the marked state amplitude amplification phenomenon which is realised using nothing more than atoms in a periodic potential which is subject to phase noise. While the system lies outside of the framework of quantum computation (making use of quantum states rather than quantum bits), the behaviour seen here may still be applicable to certain computation scenarios. (For example, it has been shown that the essence of the quantum search algorithm can be displayed in systems without qubits or entanglement [9,10]. The use of quantum states rather than quantum bits merely leads to bad resource scaling.) The Hamiltonian for the system of interest here continuously couples all of the states in question and also couples slightly to states outside the initial superposition. My surprising observation is that for such a system, applying phase noise of an appropriate strength can induce a considerable amplification of a state marked by an (adjacent) phase flip. The interesting points of this proposal are i) the generic nature of the potential used to achieve amplitude amplification and ii) the fact that noise is an essential ingredient in producing the amplification. The procedure was motivated by considering cold

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atoms exposed to a periodic potential or optical lattice. In fact, the idea represents a mere extension, if an ambitious one, of experimental techniques which have already been demonstrated in the lab [11].

Consider a two-level atom exposed to a time-independent sinusoidal potential with wave number $2k$. Rubidium or caesium atoms in a far detuned standing optical standing wave are an excellent approximation to such a system [12], with the potential induced by the AC Stark shift and the spontaneous emission rate being essentially zero on experimental times scales. The transfer of momentum momentum is discretized according to Bloch's theorem [13] and momentum eigenstates have the form of plane waves represented by states $|n2\hbar k\rangle$. We will assume that the particle has been prepared in an initial state

$$|\psi_i\rangle = \frac{1}{\sqrt{N}}(|0\rangle + |2\hbar k\rangle + \ldots - |(j-1)2\hbar k\rangle + \ldots + |(N-1)2\hbar k\rangle). \quad (1)$$

States of this sort have been considered before in the context of coherent control for kicked atoms [14]. It is in principle possible to create such a state using successive Bragg pulses in atom optical systems [15]. In this state, the $(j-1)$-th eigenstate has been flipped in phase. As we will show later, this has the effect of *marking* the j-th state for amplification. The atom thus prepared, is exposed for a time T_p to a periodic potential (with wave number $2k$) given by

$$V_t(x) = \hbar K \cos(2kx + \phi_t), \quad (2)$$

where K is the potential strength proportional to the exposure time T_p. The phase term ϕ_t changes to a new random value on the range $\phi_t \in [0, 2\pi L]$ at regular intervals T. Thus, I consider a single pulse to be made up of N individual pulse segments of length T_p/N with different random phases. The quantum evolution may be calculated by considering the evolution over each pulse segment in turn.

Here, I am interested in the external atomic dynamics during the pulse. For a pulse length T_p short compared with the time π/kv which the atoms with an rms speed v take to travel one period of the potential, it is justifiable, and convenient, to ignore the kinetic energy. To investigate these dynamics, the quantum evolution equation across an arbitrary pulse segment must be evaluated. In the short pulse approximation where contributions from momentum can be ignored (*i.e.* the Raman-Nath regime), it is given by

$$U_t = \exp(-\mathrm{i}Ht/\hbar) \approx \exp(-\mathrm{i}K\cos(2kx + \phi_t)). \quad (3)$$

Therefore, the evolution operator across all of the pulse segments is simply the product of the U_t for $t = 1, \ldots, N$. If we make the definitions $\Xi_c = (1/N)\sum_{t=1}^{N}\cos(\phi_t)$ and $\Xi_s = (1/N)\sum_{t=1}^{N}\sin(\phi_t)$ then this total evolution operator may be written

$$U_d = \exp(-\mathrm{i}K[\Xi_c\cos(2kx) + \Xi_s\sin(2kx)]). \quad (4)$$

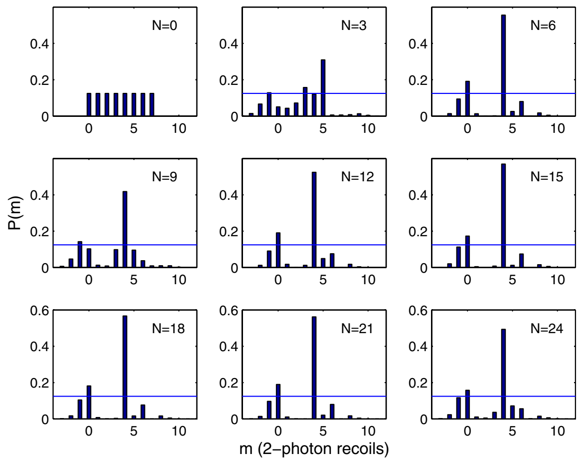

Fig. 1: (Colour on-line) The evolution of the momentum probability distribution as a function of the number of phase jumps in the pulse (shown in the upper right of each panel). The upper leftmost pannel shows the initial probability distribution.

We can find the wave function after the pulse as a function of the output integer momentum state $|m\rangle \equiv |m2\hbar k\rangle$ by taking the inner product $\psi_o(m) = \langle m|U_d|\psi_i\rangle$. Figure 1 demonstrates how the probability distribution of the integer momentum m, as given by $|\psi_o(m)|^2$, evolves as the number of phase jumps in the pulse is increased. In this case, the momentum states 0 to $7 \times 2\hbar k$ were given equal initial amplitudes and the $m = 4$ state was marked by flipping the phase of the $m = 3$ state. I took $L = 0.5$ and $K = 2.8$ (the reason for taking these values is explained below) and evolved the initial wave function by applying the quantum evolution iperator U_d for increasing numbers N of random phase jumps within the pulse. It may be seen that, as the number of phase jumps increases, the probability amplitude of the $m = 5$ state increases whilst all the other initially populated states remain approximately the same or decrease in amplitude. Certainly, no other state recieves the large amplification of the $m = 5$ state.

Observing this intriguing behaviour, two questions are immediately raised: 1) Does the same amplification behaviour occur for any choice of the marked state j? 2) What are the optimal parameters to achieve amplification? These questions are answered in figs. 2 and 3, respectively.

Firstly, in fig. 2, I plot the state of maximum amplitude after application of 30 phase jumps for the same parameters as in fig. 1 against the marked state m. As seen in fig. 2, there is perfect correlation between the initially marked state and the maximum amplitude state in the output wave function. Furthermore, as shown in the inset of fig. 2, apart from the case where the marked state corresponds to the "edge" states $|m = 0\rangle, |N-1\rangle$, the amplification

$$R = \frac{|\psi_o(j)|^2}{|\psi_i(j)|^2} = NP(j) \quad (5)$$

has the same value $R \approx 4.2$ for any marked state using this procedure, with fluctuations occuring due to the specific

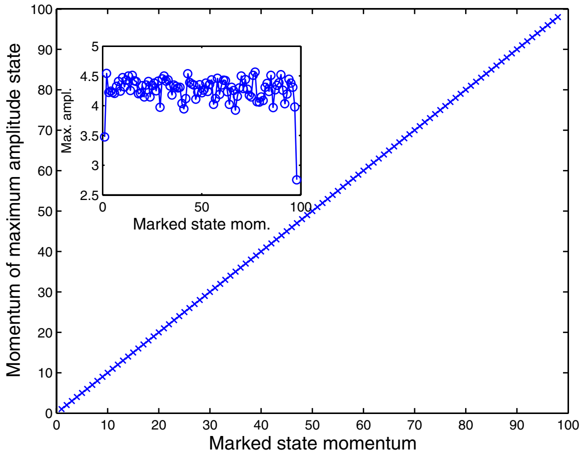

Fig. 2: (Colour on-line) Demonstration of perfect correlation between the marked state and the maximum probability state in the final output superposition. An initial superposition of 100 momentum states was used. Crosses show the number of the state which had maximum amplitude on the vertical axis, plotted against the number of the marked state in the initial superposition for a 30 phase jump pulse. We see perfect correlation between the two. The maximum amplitude state is always $|j\rangle$, where $j-1$ is the phase flipped state momentum. The inset shows the value of the maximum as a function of the initial marked state quantum number.

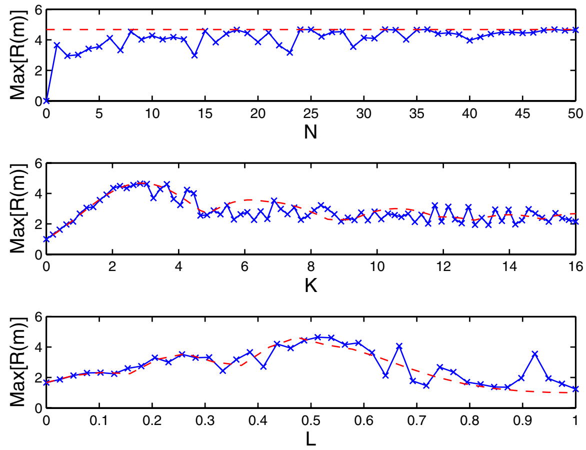

Fig. 3: (Colour on-line) Finding the optimum parameters for amplification. The upper panel shows the R parameter plotted against the number of phase jumps N in the pulse for simulations (crosses) and eq. (7) (dashed line). The same symbols are used in the middle panel which shows the same quantity as a function of the potential strength K. The bottom panel shows the variation of the normalized maximum with the noise level, demonstrating that $L=0.5$ produces the optimum amplification.

noise realization in each simulated pulse with phase noise.

Secondly, I consider the question of optimal parameters to induce amplification. There are three parameters which we can vary —the number of phase jumps N in the pulse, the over all strength of the potential K and the strength L of the phase noise. The variation of the amplification R as a function of these parameters is shown in fig. 3, where the crosses show results from quantum simulations as above. The dashed lines show results from the theory in the limit of a large number of phase jumps which is developed below. As shown in the upper panel of fig. 3, the amplification as a function of N (taking $L=0.5$ and $K=2.8$) saturates at $N\sim 20$. As a function of the potential strength (see fig. 3, middle panel), (taking $N=30$ and $L=0.5$) we find a peak at $K\sim 2.8$. Finally, measuring the amplification as a function of noise level L (with $N=30$ and $K=2.8$) gives a broad peak about $L=0.5$ (see the bottom panel of fig. 3).

The simulation results presented so far demonstrate a substantial and general amplification effect, whereby a marked state $|m=j\rangle$ in an initial coherent ensemble can be amplified by means of a sinusoidal potential with added phase noise. To understand this phenomenon better, I now develop a closed form theoretical expression for the output wave function. The results shown in the upper panel of fig. 3 suggest that the amplification saturates as the number of phase jumps is increased. Therefore, it seems reasonable to develop a theory in the limit of a large number of phase jumps per pulse.

Following this logic, I approximate the quantities Ξ_c and Ξ_s by the *continuous* mean of the cos and sin functions over ϕ. Specifically, for a large number of noise events, $\Xi_c \approx \int_0^{2\pi L}\cos(\phi)\mathrm{d}\phi = \mathrm{sinc}(2L)$ and $\Xi_s \approx \int_0^{2\pi L}\sin(\phi)\mathrm{d}\phi = (\pi L/2)\mathrm{sinc}^2(L)$, where I take the normalized definition $\mathrm{sinc}(x)=\sin(\pi x)/(\pi x)$. Thus the approximate quantum evolution operator in the limit of a large number of phase jumps is

$$\begin{aligned} U_c \approx\ & \exp[-\mathrm{i}K\mathrm{sinc}(2L)\cos(2kx)] \\ & \times\exp[-\mathrm{i}K(\pi L/2)\mathrm{sinc}^2(L)\sin(2kx)]. \end{aligned} \tag{6}$$

I now calculate the wave function after application of the pulse given the initial state 1. From refs. [11], the wave function after such a pulse for a single initial momentum state $|m\rangle$ is $\psi_o(m)=\langle m|U|0\rangle = \mathrm{i}^m(\overline{A}/A)J_m(|A|)$, where $A=K[\mathrm{sinc}(2L)+\mathrm{i}(\pi L/2)\mathrm{sinc}^2(L)]$. Using linearity, the final wave function starting from the initial state (1) is

$$\psi_o(m)=\langle m|U|\psi_i\rangle = \frac{1}{\sqrt{N}}\chi_m\left[\sum_{l=0}^{N-1}\Phi_l\chi_l J_{m-l}(|A|)\right], \tag{7}$$

where $\chi_l=\mathrm{i}^{-l}(\overline{A}/A)^{-l/2}$ and $\Phi_l=1-2\delta_{l,j-1}$. By using the Kronecker δ form of Φ_l, the wave function may be written in a more useful form:

$$\psi_o(m)=\frac{1}{\sqrt{N}}\chi_m[S_m-2M_{m-j+1}], \tag{8}$$

where $S_m=\sum_{l=0}^{N-1}\chi_l J_{m-l}(|A|)$ and $M_{m-j+1}=\chi_j J_{m-j+1}(|A|)$. The dashed lines in fig. 3 show the

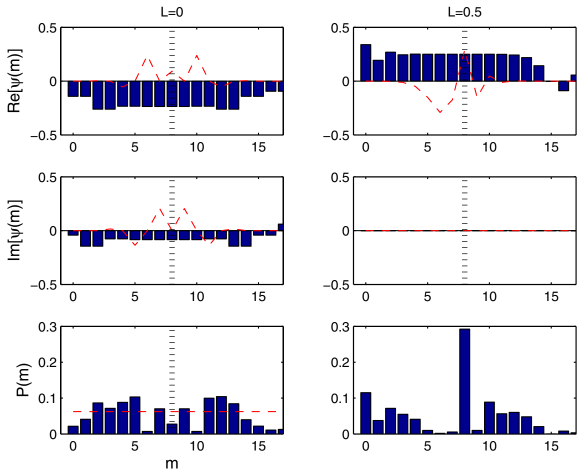

Fig. 4: (Colour on-line) Real and imaginary parts of components of the wave function (upper two panels) and the quantum probability distribution are plotted plotted for zero noise (left column) and optimal noise ($L = 0.5$) (right column) for amplification of a 16 component initial state where the $|m = 8\rangle$ state was marked by flipping the $|m = 7\rangle$ state. In both cases the upper two panels show the sum term S_m as bars and the marked state term $-2M_{m-j+1}$ term as a dashed line. The marked state is inidicated by a thick dotted line in each figure.

results of using the above theory to estimate the quantity $\max(P)$ for the same parameters as used in the simulations. There is good qualitative and fair quantitative agreement between the theory and simulations, with any differences being due to the limited number of phase jumps used in the simulations. Given this agreement, it seems reasonable to use the approximate wave function (8) to understand the properties of the system.

Equation (8) provides some illumination of the amplification procedure. Firstly, note that the quantity S_m is a constant for any m and so does not explain the amplification which must have some dependence on the marked state j. Therefore, the term M_{m-j+1} contains all the information about the amplification of a specific state. Figure 4 shows how $-2M_{m-j+1}$ varies with m. For the optimal parameters, this quantity is maximal when $m = j$, leading to amplification of the marked state (*i.e.*, the state with momentum one unit more than the phase flipped state).

At this point, the reader may well be suspicious about the roll that noise plays in the amplification scheme as explained above. The only parameter of importance for determining the exact nature of M_{m-j+1} is $|A|$. Although A depends explicitly on the noise level, it can just as well be varied by changing K, and so at first glance it may seem that noise is superfluous to the amplification scheme. However, the above argument for why amplification occurs does not consider the phase between the terms S_m and M_{m-j+1}, which is difficult to estimate from eq. (8). However, the phase is vitally important in determining whether there is constructive or destructive interference between the two terms. It is precisely because the noise can alter the phase between these two terms that it is necessary for this procedure to work. I demonstrate this fact in fig. 4 which shows the term S_m as bars for each m and the term M_{m-j+1} as a dashed line. The right-hand column shows results for the zero noise case ($L = 0$) and the right-hand column shows the results of the same procedure but for $L = 0.5$. In both columns, the top row shows the real part of the wave function, the middle row shows the imaginary part and the bottom row shows the quantum probability. Note that in the case of zero noise, and for the same effective kicking strength, there is *no amplification* of the marked state for zero phase noise. In the second column of fig. 4, where $L = 0.5$, we see that the noise induces the wave function to become purely real, and that furthermore, constructive interference occurs at the marked state due to the fact that $-2M_{m-j+1}$ is positive for $m = j$ but negative for surrounding states.

To be sure, the amplification effect *can* be achieved by using systematic, rather than noisy variation of the pulse phase. Sweeping the phase between 0 and π for example achieves the same amplification. Simulations also demonstrated that a fixed phase jump of $\pi/3$ per pulse segment can also give amplitude amplification, although the amplification quickly decays after a few phase jumps. The point however, is that the same degree of amplification can be achieved using *only* randomly distributed phase jumps. So long as there is some control over the mean phase and the phase variance (*e.g.*, through damping of vibrations or some other method), amplification can be achieved *without* systematic control of the potential phase. This is all the more interesting in light of the fact that phase noise is known to be a factor limiting the fidelity of the inversion-about-mean procedure [16]

It is further interesting to note that merely implementing a phase jump with the *average* value of the noisy phase fluctuations (*i.e.* $L = 0.5 \Rightarrow \langle\phi\rangle = \pi/2$) does not produce appreciable amplification, all other parameters being the same. This fact demonstrates that the fluctuations themselves are important in producing the amplification and differentiates this result from studies of amplitude amplification using arbitrary phases [17].

From the above analysis of eq. (8), it is clear that the phase-noise induced amplification scheme will work for any number of initial states. The procedure works by reducing the amplitude of states neighbouring the marked state and constructively adding to the marked state's amplitude. It is instructive to compare the amplification procedure here with the most famous quantum amplitude amplification algorithm —Grover's inversion-about-mean (IAM) procedure [3]. In terms of the state amplitudes a_m, the IAM procedure may be written compactly as

$$a'_m \to -a_m + 2\langle a\rangle, \tag{9}$$

where $\langle a\rangle = (1/N)\sum_m a_m$ is the mean of the initial amplitudes. In a search problem amongst N items, the initial states are intially all equally likely and hence $a_m = 1/\sqrt{N}$. If just one state j is flipped so that $a_j = -1/\sqrt{N}$, the mean

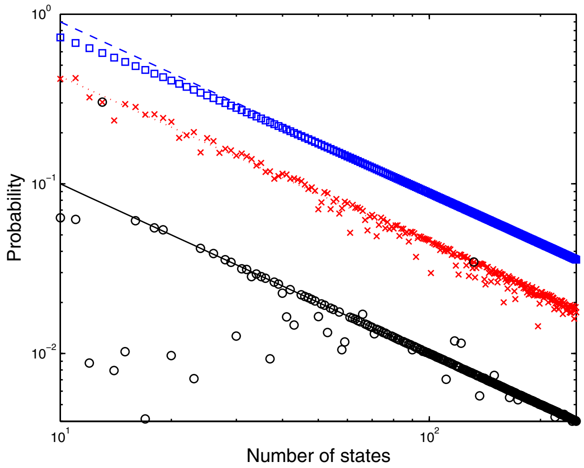

Fig. 5: (Colour on-line) The probability of measuring the marked state (which is randomly selected) is shown as a function of the number of states in the original superposition (crosses). The probability of measuring a randomly selected state is also shown (circles) and seen to lie largely on a line corresponding to $P=1/N$ (solid line). On average, independent of the number of states, a $\sim 6\,\text{dB}$ amplification of the marked state compared with a randomly selected state as can be seen by the fact that the crosses lie on the line $P\approx 4.2/N$ (dash-dotted line). The line $9/N$ is also plotted (blue dashed line) to allow comparison with the limiting maximum probability due to one iteration of the inversion-about-mean procedure. Squares show the N dependent maximum probability for IAM which is given by $9(N-1)^2/N^3$.

becomes $\langle a\rangle=(N-2)/(N\sqrt{N})$. Thus after a single iteration, the IAM procedure gives an amplification of the *probability* of measuring the j state given by

$$R_{\text{IAM}}=N(a_j')^2=9\frac{(N-1)^2}{N^2}. \tag{10}$$

Note that $R_{\text{IAM}}\to 9$ and so the amplification due to to the IAM procedure is about twice that of the best state amplification given by the phase noise method.

In fig. 5, I show how the phase-induced amplification and IAM methods compare for increasing initial state population N. Because the phase-induced amplification method relies inherently on random fluctuations, there is more random scatter associated with the amplification than for the perfectly deterministic IAM. Nonetheless, the amplification clearly clusters around the value $4.2/N$. The proability of finding a random state is also shown as circles, and clearly clusters about the expected value of $1/N$.

It is not surprising that the phase-noise–induced state amplification is not as impressive as that for IAM since Grover's method is actually the optimal possible search algorithm for polynomial resource scaling [1], and thus IAM must be the best amplification procedure available. The more disappointing aspect of the present method is that there is no obvious way to extend it to successive iterations which would give more amplification. The suceeding iteration would have to draw quantum amplitude from states more distant in terms of momentum quantum number than the currently studied method, which depletes four nearby states to amplify the marked one. It is by no means clear whether the M_{j-m+1} term can be adjusted so as to achieve this. Even if an iteration procedure can be found, it would also be necessary to apply the procedure to quantum bits rather than simple quantum states in order to produce a viable (efficient) quantum search procedure in line with the Grover algorithm.

These problems aside however, I would like to recap the interesting aspects of the present scheme. Firstly, it achieves a respectable level of quantum amplitude amplification (up to half that of the inversion-about-mean method) of a marked state using only a generic sinusoidal potential to manipulate the quantum input state. Secondly, and most remarkably, phase noise is a *fundamental* part of the amplification procedure, rather than a source of decoherence as it would be in most quantum control settings. For these reasons alone, it seems to the author that the phenomenon deserves further study.

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