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An effective action approach to photon propagation on a magnetized background

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Abstract – A new explicit analytical form of the dispersion relation for photon propagation in the presence of a strong background magnetic field is derived within the effective action framework. The dispersion relation is expressed in terms of well-known special functions, and the treatment is exact within the linearization procedure, the one-loop approximation, and the soft photon approximation. The results are incorporated in a kinetic spin plasma description for the purpose of studying quantum electrodynamical effects of strongly magnetized plasmas. The results are applied to astrophysical examples.

The response of the quantum electrodynamical (QED) vacuum to an external field is similar to that of a nonlinear medium in the sense that the vacuum acquires a polarization and a magnetization. In recent years there have been an increasing interest for QED vacuum effects. This has partly been motivated by advances in high-intensity laser technologies [1]. It is expected that already the next generation high-power laser systems will take us to intensity regimes where quantum vacuum effects may be directly observable. There are even suggested schemes for reaching the critical Sauter-Schwinger field strength [2].

QED has also found applications in astrophysics. Some observations [3] indicate that the magnetic field strength of certain types of neutron stars, called magnetars, may surpass the critical magnetic field strength B_{cr} at which the cyclotron energy equals the electron rest mass. ($B_{cr} = m_e^2/e \approx 4.4\times 10^9\,\mathrm{T}$, using natural units; $\hbar = c = 1$. Here m_e is the electron rest mass and e is the elementary charge.) Detailed knowledge of QED polarization effects in strongly magnetized plasmas can help understanding the thermal spectra observed from magnetars [4].

The purpose of this paper is twofold. First we study vacuum polarization effects on a test photon propagating on a background magnetic field of arbitrary field strength. Effects of a magnetized vacuum has been extensively studied in a vast number of publications, see, *e.g.*, [5–16]. To this field of research we contribute with a new explicit analytical form of the vacuum dispersion relation in terms of the gamma function $\Gamma(h)$, its logarithm $\ln\Gamma(h)$, the derivative of $\ln\Gamma(h)$ which is denoted $\psi(h)$ and the first derivative of the Hurwitz zeta function with respect to the first argument $\zeta'(-1,h)$ (see, *e.g.*, [17]). We have used an effective action approach, and the treatment is exact within the linearization procedure, the one-loop approximation, and the soft photon approximation. Since we include the non-transversality of the test photon in our description, our expression is valid even for ultra strong magnetic fields where the phase velocity of a test photon in the parallel mode is significantly affected. Thus, our results maps the entire region $0 \leqslant B < \infty$, provided that the one loop process gives the dominating contribution. This dispersion relation is the main result of this paper.

Secondly, we incorporate the vacuum polarization effects into a kinetic spin plasma description for the purpose of studying QED effects in strongly magnetized plasmas. This gives the results a wide range of applicability to astrophysical environments. Whereas spin statistics have previously been included in kinetic plasma models, see, *e.g.*, ref. [18], this is to our knowledge the first time that spin dynamics and the intrinsic magnetization that follows is modeled in kinetic plasmas apart from the recent paper [19] where this theory was first outlined. We apply our results to astrophysical examples.

The effective action approach to QED translates the properties of a full quantum theory into classical electrodynamics. Thus, QED polarization effects show up as

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$$\gamma_{\mathcal{F}} = -1 - \frac{\alpha}{2\pi}\left[\frac{1}{3} + 2h^2 - 8\zeta'(-1,h) + 4h\ln\Gamma(h) - 2h\ln h + \frac{2}{3}\ln h - 2h\ln 2\pi\right], \tag{3a}$$

$$\gamma_{\mathcal{FF}} = \frac{\alpha}{2\pi B^2}\left[\frac{2}{3} + 4h^2\psi(1+h) - 2h - 4h^2 - 4h\ln\Gamma(h) + 2h\ln 2\pi - 2h\ln h\right], \tag{3b}$$

$$\gamma_{\mathcal{GG}} = \frac{\alpha}{2\pi B^2}\left[-\frac{1}{3} - \frac{2}{3}\psi(1+h) - 2h^2 + (3h)^{-1} + 8\zeta'(-1,h) - 4h\ln\Gamma(h) + 2h\ln 2\pi + 2h\ln h\right], \tag{3c}$$

an additional contribution to the classical equations of motion. The one loop effective action for light propagation [20] is given by

$$\mathcal{L} = -\mathcal{F} - \frac{1}{8\pi^2}\int_0^{i\infty}\frac{\mathrm{d}s}{s^3}e^{-m_e^2 s}\left[(es)^2 ab\coth(eas)\cot(ebs) - \frac{(es)^2}{3}\left(a^2-b^2\right) - 1\right] - A_\alpha j^\alpha, \tag{1}$$

where $a = [(\mathcal{F}^2+\mathcal{G}^2)^{1/2} + \mathcal{F}]^{1/2}$ and $b = [(\mathcal{F}^2+\mathcal{G}^2)^{1/2} - \mathcal{F}]^{1/2}$. This Lagrangian is derived under the soft photon approximation, $\omega \ll m_e$. Here $\mathcal{F} = \frac{1}{4}F_{ab}F^{ab} = \frac{1}{2}\left(\mathbf{B}^2 - \mathbf{E}^2\right)$, $\mathcal{G} = \frac{1}{4}F_{ab}\widehat{F}^{ab} = -\mathbf{E}\cdot\mathbf{B}$, where F^{ab} is the electromagnetic field tensor, $\widehat{F}^{ab} = \epsilon^{abcd}F_{cd}/2$, ϵ^{abcd} is the totally antisymmetric tensor, A_α is the four-potential and j^α is the four-current. Note that for a pure magnetic field, we have $\mathcal{F} = B^2/2$ and $\mathcal{G} = 0$.

The equations of motion are obtained using Euler-Lagrange equations, and takes the form [6,14]

$$\gamma_{\mathcal{F}}\partial_\mu F^{\mu\nu} + \frac{1}{2}\left[\gamma_{\mathcal{FF}}F^{\mu\nu}F_{\alpha\beta} + \gamma_{\mathcal{GG}}\widehat{F}^{\mu\nu}\widehat{F}_{\alpha\beta} + \gamma_{\mathcal{FG}}\left(F^{\mu\nu}\widehat{F}_{\alpha\beta} + \widehat{F}^{\mu\nu}F_{\alpha\beta}\right)\right]\partial_\mu F^{\alpha\beta} = -j^\nu, \tag{2}$$

where we have used the notation $\gamma_{\mathcal{F}} = \partial\mathcal{L}/\partial\mathcal{F}$, $\gamma_{\mathcal{FF}} = \partial^2\mathcal{L}/\partial\mathcal{F}^2$, etc.

If no external electric field is present, the scalars $\gamma_{\mathcal{F}}$, $\gamma_{\mathcal{G}}$, $\gamma_{\mathcal{FF}}$, $\gamma_{\mathcal{GG}}$ and $\gamma_{\mathcal{FG}}$ can be calculated analytically. To do this, we express the derivatives $\partial_{\mathcal{F}}$ and $\partial_{\mathcal{G}}$ in terms of a and b; $\partial_{\mathcal{F}} = \left(a^2+b^2\right)^{-1}(a\partial_a - b\partial_b)$ and $\partial_{\mathcal{G}} = \left(a^2+b^2\right)^{-1}(b\partial_a + a\partial_b)$. We then perform the differentiation on the terms inside the square bracket of (1), after which we take the limit $b \to 0$. The remaining integrals can be solved analytically using regularization techniques [21]. We note that convergence of the integrals for $0 < h < \infty$ is implicitly ensured by the prescription $m_e^2 \to m_e^2 - i\epsilon$.

The scalars $\gamma_{\mathcal{F}}$, $\gamma_{\mathcal{G}}$, $\gamma_{\mathcal{FF}}$, $\gamma_{\mathcal{GG}}$ and $\gamma_{\mathcal{FG}}$ take on the following expressions: $\gamma_{\mathcal{G}} = 0$, $\gamma_{\mathcal{FG}} = 0$,

see eqs. (3a)–(3c) *above*

where $\alpha = e^2/4\pi$ is the fine structure constant, and we have defined $z = esa$ and $h = m_e^2/2ea = B_{cr}/2B$. For weak fields, the scalars $\gamma_{\mathcal{F}}$, $\gamma_{\mathcal{FF}}$ and $\gamma_{\mathcal{GG}}$ agrees with the results we get if the well-known weak field expansion of eq. (1) were used:

$$\mathcal{L}_w = -\mathcal{F} + \frac{\alpha}{90\pi}\frac{1}{B_{cr}^2}(4\mathcal{F}^2 + 7\mathcal{G}^2) - A_\alpha j^\alpha. \tag{4}$$

Furthermore, $\nabla^2\mathcal{L} = \gamma_{\mathcal{FF}} + \gamma_{\mathcal{GG}}$ agrees analytically with the result found in eq. (3.48) of ref. [21].

A legitimate question is how radiative corrections will alter the equations for ultra strong fields. Starting from some basic assumptions of how higher-loop corrections scales with the field strength, it can be argued that these corrections are likely to be harmless [21]. Thus, the physical response of the system may be largely governed by the dynamics described by the one-loop Lagrangian even for ultrastrong magnetic fields. So, with the use of the scalar expressions (3a)–(3c) it is possible to study light propagation effects at arbitrary magnetic field strengths.

Next we will derive a dispersion relation for light propagation and introduce a kinetic spin plasma description to model the current j^ν. We assume that a weak electromagnetic field is propagating on a background of a strong external magnetic field, such that $F_{tot}^{\mu\nu} \to F^{\mu\nu} + f^{\mu\nu}$, where $F^{\mu\nu}$ is a strong static and isotropic field and $f^{\mu\nu}$ is a weak field that has a harmonic oscillation $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Linearizing the equations of motion, $\partial_\mu \to -ik_\mu$, $\gamma_{\mathcal{F}} \to [\partial\mathcal{L}/\partial\mathcal{F}]_{F_{tot}^{\mu\nu}=F^{\mu\nu}}$, the space component of eq. (2) can be written as

$$\gamma_{\mathcal{FF}}\left(\mathbf{B}_0\cdot(\mathbf{k}\times\mathbf{E}_1)\right)(\mathbf{k}\times\mathbf{B}_0) - \gamma_{\mathcal{GG}}\omega^2\mathbf{B}_0\left(\mathbf{B}_0\cdot\mathbf{E}_1\right) + \gamma_{\mathcal{F}}\left(\omega^2\mathbf{E}_1 - k^2\mathbf{E}_1 + \mathbf{k}\left(\mathbf{k}\cdot\mathbf{E}_1\right)\right) = i\omega\mathbf{j}, \tag{5}$$

where we have used $\nabla\times\mathbf{E}_1 = -\partial_t\mathbf{B}_1$, and where the subindex 0/1 denotes field components of the background/weak field, respectively.

We use the kinetic spin plasma theory outlined in ref. [19] to obtain an expression for the current vector,

$$\mathbf{j} = \mathbf{j}_{\mathrm{free}} + \nabla\times\mathbf{M} = \sum_i\left[q_i\int\mathbf{v}f_i\mathrm{d}\Omega + 2\mu_i\nabla\times\int\mathbf{s}f_i\mathrm{d}\Omega\right], \tag{6}$$

where $\mathbf{M}$ is the magnetization, $f(\mathbf{r},\mathbf{v},\mathbf{s},t)$ is the distribution function, $\mathbf{s}$ is the spin operator with norm $|\mathbf{s}| = 1/2$, $\mu_i \approx q_i/m_i$ is the magnetic moment of particle species i with charge q_i, and $\mathrm{d}\Omega = v_\perp\mathrm{d}v_\perp\mathrm{d}\varphi_v\mathrm{d}v_z\sin\theta_s\,\mathrm{d}\theta_s\,\mathrm{d}\varphi_s$ with the velocity expressed in cylindrical coordinates and the

$$D^{ij}=\begin{pmatrix} -\gamma_{\mathcal{F}}\left(\omega^2-k_\parallel^2\right) & 0 & -\gamma_{\mathcal{F}}k_\perp k_\parallel \\ 0 & -\gamma_{\mathcal{F}}\left(\omega^2-k^2\right)+\gamma_{\mathcal{F}\mathcal{F}}k_\perp^2 B_0^2+\omega_p^2\frac{k_\perp^2k_\parallel^2}{4m^2\omega^2}\left(\frac{\omega^2}{k_\parallel^2}-1\right) & -i\omega_p^2\frac{k_\perp k_\parallel}{2m\omega} \\ -\gamma_{\mathcal{F}}k_\perp k_\parallel & i\omega_p^2\frac{k_\perp k_\parallel}{2m\omega} & -\gamma_{\mathcal{F}}\left(\omega^2-k_\perp^2\right)+\gamma_{\mathcal{G}\mathcal{G}}\omega^2B_0^2-\omega_p^2 \end{pmatrix}, \quad (11)$$

spin expressed in spherical coordinates. Whereas our QED description of the vacuum remain valid for arbitrary field strengths, we will for simplicity restrict the spin plasma description to be valid for strong magnetic fields far from resonance, such that $\omega_c = qB/m \gg \omega$. Under such conditions, the Vlasov equation reduces to

$$\partial_t f + \mathbf{v}\cdot\nabla f + \left[\frac{q}{m}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) + \frac{2\mu_i}{m}\nabla\left(\mathbf{s}\cdot\left(\mathbf{B}-\mathbf{v}\times\mathbf{E}\right)\right)\right]\cdot\nabla_v f = 0. \quad (7)$$

Here we have included the spin-orbit coupling which for strong magnetic fields gives a contribution of the same order as the standard spin terms. We may write the distribution function as $f(\mathbf{r},\mathbf{v},\mathbf{s},t)=f_0(\mathbf{v},\mathbf{s})+f_1(\mathbf{r},\mathbf{v},\mathbf{s},t)$, where f_0 is the Fermi-Dirac equilibrium distribution for thermal equilibrium and for a large chemical potential,

$$f_0 = n_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{\mu_iB_0\exp\left[-\frac{mv^2/2+\mu_iB_0\cos\theta_s}{k_BT}\right]}{4\pi k_BT\sinh\left(\frac{\mu_iB_0}{k_BT}\right)}. \quad (8)$$

f_0 is normalized such that $\int f_0 \mathrm{d}\Omega = n_0$ with n_0 being the number density. Here we have without loss of generality defined $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ and $k_y=0$. If we linearize eq. (7) we can solve this new equation by making a plane wave ansatz, $f_1=\tilde{f}_1\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]$, and write $\tilde{f}_1$ as an expansion in eigenfunctions $\psi_a(\varphi_v,v_\perp)=(2\pi)^{-1/2}\times\exp[-i(a\varphi_v-k_\perp v_\perp\sin\varphi_v/\omega_c)]$ such that

$$\tilde{f}_1=\frac{1}{\sqrt{2\pi}}\sum_{a,b}g_{ab}(v_\perp,v_z,\theta_s)\psi_a(\varphi_v,v_\perp)\exp(-ib\varphi_s), \quad (9)$$

where $a=0,\pm1,\pm2,\ldots$ and $b=-1,0,1$. By recognizing that $\int_0^{2\pi}\psi_a\psi_b^*\mathrm{d}\varphi_v=\delta_{ab}$ it is straightforward to show from the linearized version of eq. (7) that the most contributing terms to eq. (9) under the assumption $\omega_c\gg\omega$ are the terms where $a,b=0$. We get $i(\omega-k_zv_z)g_{00}=2\pi I_{00}$ where

$$I_{00}\approx\frac{q}{m}E_{1z}+i\frac{\mu_i}{m}\cos\theta_s\left[B_{1z}k_z\frac{\partial f_0}{\partial v_z}-\frac{k_xE_{1y}}{2}v_\perp\frac{\partial f_0}{\partial v_\perp}\right] \quad (10)$$

assuming $k_\perp^2k_BT/m\ll\omega_c^2$. We treat ions as immobile, neglect their spin contribution and perform the integrals in eq. (6). The result is inserted in eq. (5). The system equations can now be written as $D^{ij}E_{1j}=0$, where

see eq. (11) *above*

where $\omega_p^2=q_e^2n_0/m_e$ is the plasma frequency. The determinant of D^{ij} gives us the dispersion relation. The classical plasma contribution is found in the D_{33}-term, while the other plasma contributions originates from spin dynamics. Here we have assumed $\mu_eB_0\gg k_BT$ which is well satisfied already for standard neutron star parameters ($T\sim10^6$ K, $B\gtrsim10^8$ T) [22]. In the analysis below, we will first focus on pure vacuum effects, and later discuss spin plasma effects.

In a magnetized vacuum, we have two normal modes present: the orthogonal mode where the polarization is orthogonal to the $\mathbf{B}_0,\mathbf{k}$-plane, and the parallel mode where the polarization lies in that plane. The phase velocity $\nu=\omega/k$ for the two modes takes the form

$$\nu_{\perp\mathrm{vac}}^2=1+\frac{\gamma_{\mathcal{F}\mathcal{F}}B_0^2}{\gamma_{\mathcal{F}}}\sin^2\theta_B, \quad (12a)$$

$$\nu_{\parallel\mathrm{vac}}^2=\frac{1-\frac{\gamma_{\mathcal{G}\mathcal{G}}B_0^2}{\gamma_{\mathcal{F}}}\cos^2\theta_B}{1-\frac{\gamma_{\mathcal{G}\mathcal{G}}B_0^2}{\gamma_{\mathcal{F}}}}, \quad (12b)$$

where θ_B is the angle between $\mathbf{B}_0$ and $\mathbf{k}$. For weak fields, eqs. (12a) and (12b) reduces to the usual vacuum birefringence expressions; $\nu_\perp\approx1-(8\alpha^2B_0^2/45m^4)\sin^2\theta_B$ and $\nu_\parallel\approx1-(14\alpha^2B_0^2/45m^4)\sin^2\theta_B$. Vacuum birefringence in strong magnetic field regimes have been studied in several previous publications; *e.g.*, ref. [7] approaches the problem analytically using the vacuum polarization tensor, and, *e.g.*, ref. [8] has studied the problem numerically for various photon energies. An effective action approach is used by ref. [9] who expressed the Lagrangian (1) in terms of special functions, and ref. [11] expresses the Lagrangian as a non-perturbative slowly convergent series expansion [23]. Reference [24] uses the same approach as ref. [11] but also takes non-transversality of the photon into account.

In our derivation of eqs. (12a) and (12b), we have not neglected the non-transversal behavior of the photon. Thus, the results are valid for arbitrary magnetic field strengths. Vacuum birefringence for orthogonal propagation ($\theta_B=\pi/2$) of the orthogonal and the parallel mode is illustrated in fig. 1. The results are found to agree (numerically) with that of ref. [24] if a sufficiently large number

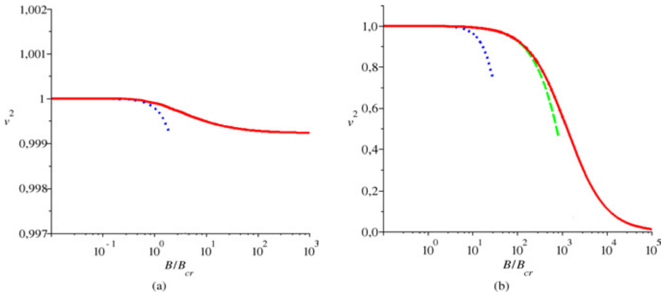

Fig. 1: (Color online) The phase velocity squared for the orthogonal (a) and the parallel mode (b) as a function of B/B_{cr} at orthogonal propagation. The dashed line is the results found in, *e.g.*, ref. [7] and the dotted line is the weak field result of eq. (4). The deviation in (a) between the result of ref. [7] and our result is smaller than the resolution of the figure.

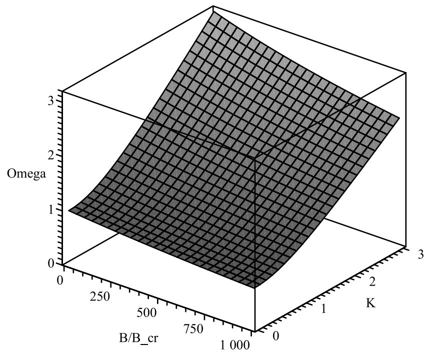

Fig. 2: Ω as a function of K for increasing values of B/B_{cr} for orthogonal propagation of the parallel mode.

terms in their series are calculated (≈ 20000 for the highest field strengths). It is evident that in the limit of an infinite strong magnetic field, propagation in the parallel mode is strictly forbidden, whereas the orthogonal mode is only slightly affected by the vacuum polarization. Photons in the parallel mode will consequently be forced to follow the magnetic field lines. Already for field strengths of a few hundred B_{cr}, we may expect a significant magnetic lensing effect. As a result, a surface area of a neutron star measured by the two polarization modes will be different.

Now we will include spin plasma effects in the analysis. We have seen that the QED effects are small for all field strengths in the orthogonal mode. Consequently, we neglect the QED contribution to this mode and restrict our analysis to include spin effects. If we assume orthogonal propagation ($k_{\|} = 0$), the dispersion relation becomes

$$v^2_{\perp\text{plasma}} = 1 - \omega_p^2/4m^2. \tag{13}$$

For magnetar crust densities, $\rho \sim 10^9\,\text{kg/m}^3$ [22], we get $\omega_p^2/4m^2 \sim 10^{-3}$.

For orthogonal propagation in the parallel mode the spin effects do not contribute to the dispersion relation. The dynamics is instead governed by QED and classical plasma effects. The dispersion relation takes the form

$$\Omega_{\|} = \left[\left(1 - \gamma_{\mathcal{F}} K^2\right) / \left(-\gamma_{\mathcal{F}} + \gamma_{\mathcal{GG}} B_0^2\right)\right]^{1/2}. \tag{14}$$

Here we have normalized the relevant parameters according to $\Omega = \omega/\omega_{pe}$ and $K = k/\omega_{pe}$. This wave mode is classically not affected by the presence of a magnetic field, so the magnetic field dependence in fig. 2 is a pure QED effect. At field strengths around $300B_{cr}$, the cut-off frequency is reduced by $\sim 10\%$. At high frequencies, $\omega_p \ll \omega$, plasma effects are small and the dynamics can be modeled by eq. (12b).

In this paper we have used a recently developed spin plasma model [19] to capture the spin dynamics. In order to further establish our results, it is of interest for future research to extend this model to also include strongly relativistic effects and effects from, *e.g.*, Landau quantization [25]. While our plasma treatment here is only valid for strong magnetic fields, our analytic dispersion relation for photon propagation in vacuum is valid for arbitrary strong background magnetic field strengths. This dispersion relation is the main result of this paper and it is exact within the one-loop approximation, the linearization procedure and the soft photon approximation.

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