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To cite this article: S. Chakraborty *et al* 2010 *EPL* **91** 40007

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The generalized second law of thermodynamics and the nature of the entropy function

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received 28 June 2010; accepted in final form 11 August 2010

published online 15 September 2010

PACS 05.70.-a – Thermodynamics

PACS 95.30.Tg – Thermodynamic processes, conduction, convection, equations of state

PACS 98.80.Cq – Particle-theory and field-theory models of the early Universe (including cosmic pancakes, cosmic strings, chaotic phenomena, inflationary universe, etc.)

Abstract – In black-hole physics, the second law of thermodynamics is generally valid whether the black hole is a static or a non-static one. Considering the universe as a thermodynamical system the second law of black-hole dynamics extends to the non-negativity of the sum of the entropy of the matter and the horizon, known as generalized second law of thermodynamics (GSLT). Here, we have assumed the universe to be bounded by the event horizon where Bekenstein entropy-area relation and Hawking-temperature are not applicable. Thus considering entropy to be an arbitrary function of the area of the event horizon, we have tried to find the nature of the entropy function for the validity of the GSLT both in the quintessence era and in the phantom era. Finally, some graphical representation of the entropy function has been presented.

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In black-hole physics, the semi-classical description shows that just like a black body, black hole emits thermal radiation (known as Hawking radiation) and it completes the missing link between a black hole and a thermodynamical system. The temperature (known as the Hawking temperature) and the entropy (known as Bekenstein entropy) are proportional to the surface gravity at the horizon and area of the horizon [1,2], respectively (*i.e.* related to the geometry of the horizon). Also this temperature, entropy and mass of the black hole satisfy the first law of thermodynamics [3]. As a result, physicists start speculating about the relationship between the black-hole thermodynamics and Einstein's field equations (describing the geometry of space time). It is Jacobson [4] who first derived Einstein field equations from the first law of thermodynamics: $\delta Q = T dS$ for all local Rindler causal horizons with δQ and T as the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon. Then Padmanabhan [5] was able to formulate the first law of thermodynamics on the horizon, starting from Einstein equations for a general static spherically symmetric space time. The following nice

equivalence:

Laws of thermodynamics \Leftrightarrow Analogous laws of black-hole dynamics (Semi-classical analysis)

\Leftrightarrow Einstein field equations (gravity theory) (classical treatment)

perhaps shows the strongest evidence for a fundamental connection between quantum physics and gravity.

Subsequently, this identity between Einstein equations and thermodynamical laws has been applied in the cosmological context considering the universe as a thermodynamical system bounded by the apparent horizon (R_A). Using the Hawking temperature $T_A = \frac{1}{2\pi R_A}$ and Bekenstein entropy $S_A = \frac{\pi R_A^2}{G}$ at the apparent horizon, the first law of thermodynamics (on the apparent horizon) is shown to be equivalent to Friedmann equations [6] and the generalized second law of thermodynamics (GSLT) is obeyed at the horizon. Also in higher-dimensional space time the relation was established for gravity with Gauss-Bonnet term and for the Lovelock gravity theory [7–9].

But difficulty arises if we consider the universe to be bounded by the event horizon. First of all, in the usual standard big-bang model the cosmological event horizon does not exist. However, the cosmological event

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horizon separates from that of the apparent horizon only for the accelerating phase of the universe (dominated by dark energy). Further, Wang *et al.* [10] have shown that both the first and second law of thermodynamics break down at the event horizon, considering the usual definition of temperature and entropy as in the apparent horizon. According to them the applicability of the first law of thermodynamics is restricted to the nearby states of local thermodynamic equilibrium while the event horizon reflects the global features of space time. Also due to the existence of the cosmological event horizon, the universe should be non-static in nature and as a result the usual definition of the thermodynamical quantities on the event horizon may not be as simple as in the static space time. They have considered the universe bounded by the apparent horizon as a Bekenstein system as Bekenstein's entropy-mass bound: $S \leq 2\pi R_A$ and entropy-area bound: $S \leq \frac{A}{4}$ are valid in this region. These Bekenstein bounds are universal in nature and all gravitationally stable special regions with weak self-gravity satisfy Bekenstein bounds. Finally, they have argued that as the event horizon is larger than the apparent horizon, so the universe bounded by the event horizon is not a Bekenstein system.

In the literature, there are lot of works [11–18], dealing with the thermodynamics of the universe bounded by the apparent horizon as it is a Bekenstein system. On the other hand, due to the above complicated nature of the thermodynamical system: the universe bounded by the event horizon (UBEH), there are few works related to it. Recently, Mazumder *et al.* [19,20], starting from the first law of thermodynamics, have examined the validity of the GSLT which states that the time variation of the sum of the entropy of the horizon (S_H) and the entropy of the matter inside it (S_I) should be positive definite, *i.e.*, $\frac{d}{dt}(S_H + S_I) \geq 0$. Without assuming any specific choice for the entropy and the temperature on the event horizon, they are able to show the validity of the GSLT with some restrictions on the matter. In the present work, we try to speculate the nature of the entropy function on the event horizon assuming the validity of the GSLT. It should be noted that the reason to stress the validity of the GSLT is that it is a universal law, it holds in any generality, irrespective of whether the thermodynamical system is an equilibrium or a non-equilibrium one.

We start with homogeneous and isotropic FRW model of the universe having line element

$$ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 d\Omega_2^2, \quad (1)$$

where $\tilde{r} = ar$ is the area radius, $h_{ab} = \text{diag}(-1, \frac{a^2}{1-kr^2})$ with $k = 0, \pm 1$ for a flat, closed and open model and $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric on the unit 2-sphere. The Friedmann equations are

$$H^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3} \quad (2)$$

and

$$\dot{H} - \frac{k}{a^2} = -4\pi G(\rho + p) \quad (3)$$

with the energy conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (4)$$

The apparent horizon, a null surface is characterized by

$$h^{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0$$

and hence the radius of the apparent horizon has the expression

$$R_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}. \quad (5)$$

On the other hand, the radius of the cosmological event horizon (which exists for the accelerating model of the universe in Einstein gravity) is given by

$$R_E = a \int_t^\infty \frac{dt}{a} = a \int_a^\infty \frac{da}{Ha^2}. \quad (6)$$

As

$$R_H = \frac{1}{H} \quad (7)$$

is the radius of the Hubble horizon, so depending on the curvature, the horizons are related by the following relations:

I) $k = 0$:

$$R_A = \frac{1}{H} = R_H < R_E;$$

II) $k = -1$:

$$R_H < R_A < R_E;$$

III) $k = +1$:

$$\begin{aligned} &\text{either } R_A < R_E < R_H, \\ &\text{or } R_A < R_H < R_E. \end{aligned}$$

In refs. [19,20], Mazumder *et al.* have assumed the validity of the first law of thermodynamics at the event horizon (*i.e.*, the Clausius relation $-dE = T_E dS_E$) where the amount of energy crossing the event horizon during the infinitesimal time dt is given by

$$-dE = 4\pi R_E^3 H(\rho + p) dt. \quad (8)$$

Here the change in the horizon entropy during the small time dt is

$$dS_E = \frac{4\pi R_E^3 H}{T_E} (p + \rho) dt, \quad (9)$$

where (S_E, T_E) are the entropy and temperature of the event horizon and (ρ, p) are energy density and thermodynamic pressure of the fluid bounded by the event horizon.

Now, to calculate the variation of the matter entropy we shall use Gibb's equation [21]

$$T_E dS_I = dE_I + p dV, \quad (10)$$

where S_I and E_I are the entropy and the energy of the matter distribution, respectively. One may note that for thermodynamical equilibrium, the temperature of the matter is chosen as that of the event horizon (*i.e.*, T_E). As

$$V = \frac{4}{3}\pi R_E^3, \quad E_I = \frac{4\pi}{3}\rho R_E^3, \quad (11)$$

then from Gibb's equation (10) and using the energy conservation relation (4) the variation of matter entropy has the expression

$$dS_I = 4\pi R_E^2 (\rho + p) \left(\dot{R}_E - H R_E \right) dt \frac{1}{T_E} \quad (12)$$

Differentiating the relation (6) for R_E , we get

$$\dot{R}_E = (H R_E - 1). \quad (13)$$

Hence combining eqs. (9) and (12) using (13), the time variation of the total entropy is given by

$$\frac{d}{dt} (S_E + S_I) = 4\pi (\rho + p) \frac{R_E^2 H}{T_E} \left(R_E - \frac{1}{H} \right). \quad (14)$$

Based on the above calculation Mazumder *et al.* [19,20] have obtained the following restrictions for the validity of the GSLT (*i.e.*, $\frac{d}{dt} (S_E + S_I) \geq 0$).

- i) For a flat and open FRW universe the GSLT is valid if the weak energy condition $\rho + p > 0$ is satisfied.
- ii) For a closed model, the validity of the GSLT demands either the weak energy condition is satisfied and $R_A < \frac{1}{H} < R_E$ or the weak energy condition is violated and $R_A < R_E < \frac{1}{H}$.
- iii) For the validity of GSLT, no specific form of entropy or temperature on the event horizon is needed.

Here in this paper, the thermodynamical study is rather in the opposite way. We start with the validity of the GSLT and infer about the properties of the entropy function. In analogy with Bekenstein's entropy-area relation, we assume the functional form of the entropy at the event horizon be

$$S_E = \frac{f(A)}{4G} \quad (15)$$

with $A = 4\pi R_E^2$, the area of the event horizon. So,

$$\frac{dS_E}{dt} = \frac{f'(A)}{G} 2\pi R_E \dot{R}_E, \quad (16)$$

where the "dash" denotes differentiation with respect to "A". Thus using the expression (12) for the variation of matter entropy, the time variation of the total entropy is given by

$$\frac{d}{dt} (S_E + S_I) = 2\pi R_E \left[\frac{(H R_E - 1)}{G} f'(A) - \frac{2R_E}{T_I} (\rho + p) \right], \quad (17)$$

where T_I is the temperature of the matter distribution and we have $T_I = T_E$ for the equilibrium thermodynamics.

Now we shall examine the validity of the GSLT in (a) the quintessence era and (b) the phantom era.

(a) In the quintessence era, the weak energy condition ($\rho + p > 0$) is satisfied and so according to Davies [22] $\dot{R}_E > 0$, *i.e.* $R_E > \frac{1}{H}$. So from eqs. (12) (using (13)) and (16) the horizon entropy will increase provided $f'(A) > 0$ while the matter entropy is decreasing with time. Thus the GSLT will be satisfied provided the expression within the square bracket in eq. (17) is positive.

(b) On the other hand, in the phantom era, there is a violation of the weak energy condition ($\rho + p < 0$) and Sadjadi [23] $\dot{R}_E < 0$, *i.e.* $R_E < \frac{1}{H}$. So the matter entropy will always increase and the horizon entropy will increase or decrease depending on $f'(A) < 0$ or $f'(A) > 0$ and as before the expression within the square bracket in eq. (17) should be positive for the validity of the GSLT.

Therefore, for the validity of the GSLT, the entropy function $f(A)$ must have the following characteristics:

(a) In the quintessence era $-f(A)$ is an increasing function of A , *i.e.*, R_E such that

$$f'(A) > \frac{2R_E G (\rho + p)}{T_I (H R_E - 1)}. \quad (18)$$

(b) In the phantom era either $f(A)$ is still an increasing function of A with

$$0 < f'(A) < \frac{2R_E G}{T_I} \left| \frac{(\rho + p)}{H R_E - 1} \right|. \quad (19)$$

or $f(A)$ is decreasing function of A , *i.e.* $f'(A) < 0$ and the GLST is identically satisfied in that era.

To have smooth entropy function across the phantom barrier one must have $f'(A) = 0$ on the barrier. As a result both matter and horizon entropy becomes constant on the phantom crossing. Thus the entropy function has either of the two possible following behaviors:

I. The entropy function increases sharply in the quintessence era so that the expression within the square bracket in eq. (17) must have positive value to satisfy the GLST and then $f(A)$ reaches a maximum at the phantom crossing, subsequently slowly decreases in the phantom era so that GSLT is satisfied there. We speculate that the graph of the entropy function throughout the quintessence and phantom era will be as in fig. 1 with a maximum at phantom crossing.

II. The entropy function increases in both the quintessence and phantom era with a point of inflexion at the phantom crossing as shown in fig. 2.

Now we shall examine whether the Bekenstein entropy-area relation is valid on the event horizon. From the inequalities (18) and (19) to have the GSLT on the event horizon with Bekenstein entropy, the temperature of the matter distribution should satisfy

$$T_I > \text{ or } < 2R_E G \left| \frac{(\rho + p)}{H R_E - 1} \right|,$$

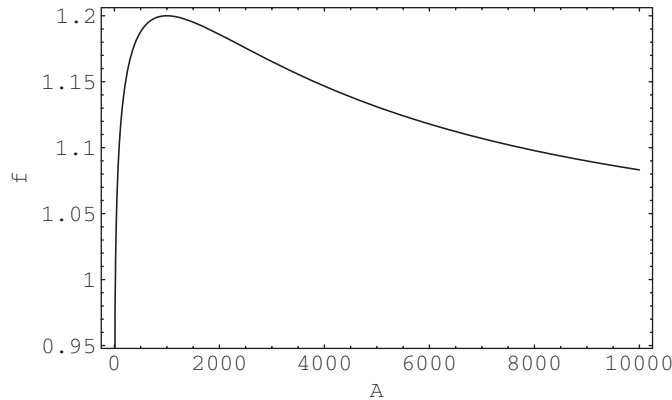


Fig. 1: The variation of the entropy function with the area function where the entropy function attains the maximum value at the phantom crossing.

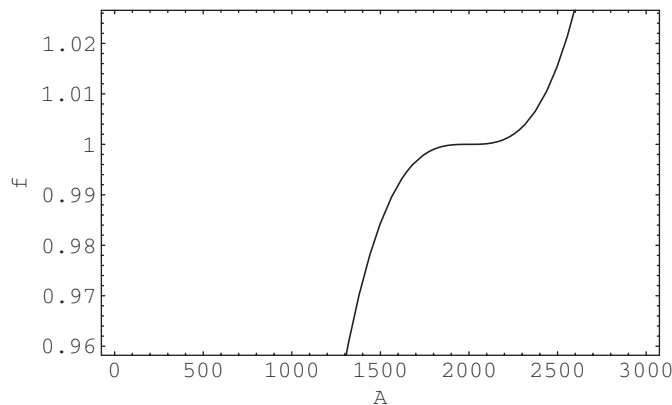


Fig. 2: The variation of the entropy function with the area function; a point of inflexion can be found at the phantom crossing.

in the quintessence or phantom era, respectively. This temperature bound is quite distinct from Hawking temperature and we have non-equilibrium thermodynamics. Finally, one may note that throughout the calculation no specific Einstein field equations have been used, only the equation of continuity is needed to calculate the variation of matter entropy. Therefore, the above result is true in any gravity theory.

RB wants to thank West Bengal State Govt, India for awarding JRF.

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