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Linear superpositions of nonlinear matter waves in optical lattices

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Abstract – It is shown that by a proper design of the nonlinearity it is possible to obtain linear superposition of matter waves in optical lattices. In particular, the possibility to create non-stationary states of Bose-Einstein condensates which are linear superposition of stationary nonlinear matter waves is demonstrated. This is achieved by means of spatial variation of the interatomic interaction which suppresses the nonlinear overlapping terms, which otherwise would destroy the superposition, and at the same time retaining all the nonlinearity necessary for each component state to exist. The superposition state is shown to be long lived and can be split into constituent parts by accelerating the lattice.

It is well known that the superposition principle, a key postulate of quantum mechanics, when applied to the macroscopic scale leads to bizarre paradoxes such as the one discussed by Schrödinger in [1] to illustrate the fuzzy correspondence between micro- and macro-worlds. It is also well known, that for interacting quantum many-body systems, like for example a diluted gas of bosons in an optical lattice (OL), the Heisenberg equations of motion for the field operators are nonlinear [2], this leading in the mean-field approximation to a nonlinear Schrödinger equation (NLSE), also known as Gross-Pitaevskii equation (GPE) [3], for the time evolution of the ground state of a Bose-Einstein condensate (BEC). Thus, the superposition principle is lost in the step from quantum to classical behavior achieved by averaging out the quantum fluctuations in the system. The nonlinearity of the classical equations well correlates with the idea of Einstein about macrorealism according to which macroscopic objects should follow classical laws, this ruling out for them the possibility to be in superposition states.

Macrorealism, however, is presently questioned by a series of experiments showing that macroscopic objects, such as a micrometer-sized superconducting circuit, can exist in linear superpositions of distinct states, following apparently the rules of quantum mechanics [4]. It should be remarked, however, that the existence of macroscopic superpositions does not necessarily contrasts with the laws of classical mechanics, this being particularly true for systems for which a superposition principle can be made to exist even in the presence of nonlinearity.

The aim of this letter is to investigate such a possibility by taking as an example the case of a BEC loaded in an OL. More specifically, we show that by allowing spatial periodic modulations of the interaction (scattering length) in the GPE it is possible to create linear superpositions of matter waves which are stable on a very long time scale. The nonlinearity modulation can be achieved by means of the optical Feshbach resonance technique [5] providing what is known as nonlinear optical lattice (NOL). NOLs have been recently realized in experiments [6] and have been theoretically shown to lead to a series of interesting phenomena in the mean-field nonlinear regime, such as long-living Bloch oscillations [7], dynamical localization [8] and Rabi oscillations [9] of gap solitons (GSs).

Spatial variations of the nonlinearity have also been used to study adiabatic changes of bright and dark solitons in parabolic traps [10], existence and stability of solitons in pure NOLs [11] and in combined linear and nonlinear lattices [12], mathematical properties of the ground state [13] and stabilization of higher modes in a

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parabolic trap [14], Lie symmetries and canonical transformations mappings to homogeneous nonlinear Schrödinger equation [15], delocalizing transitions in one-dimensional OL [16], soliton stabilization in presence of linear OLs [12,17]. The success of these investigations suggests to use NOLs to eliminate (or strongly suppress) the interaction between two different states to be put in superposition, while keeping the nonlinearity necessary for each component state to exist. We show that when this condition is realized the superposition state displays the typical interference term and becomes stable on a very long time scale.

We use as a model equation for a BEC in a linear π-periodic OL $-V\cos(2x)$, in the presence of a time-dependent force $\mathcal{F}(t)$ (acceleration of the OL) and a space-dependent nonlinearity $\mathcal{G}(x)$, the following one-dimensional GPE written in dimensionless form as

$$i\frac{\partial\psi}{\partial t}=-\frac{\partial^2\psi}{\partial x^2}-V\cos(2x)\psi+\mathcal{F}(t)x\psi+\mathcal{G}(x)|\psi|^2\psi. \quad (1)$$

Here energies (like OL amplitude V) are measured in units of the recoil energy $E_R=\hbar^2\pi^2/(2md^2)$ (where m is the atomic mass and d is the OL period), while the space x and time t are measured in units of d/π and $\hbar/E_R$, respectively. The wave function $\psi(x)$ is normalized to $a_\perp^2\pi^2/(4d^2|a_s|)$, with $a_\perp$ the transverse (in the (y,z)-plane) oscillator length and a_s the average scattering length. Besides BEC, eq. (1) is also a well-known model for the propagation of light in periodically modulated media (see [18] for a recent review) and for the propagation of weak light beams through a gas of three level atoms [19]. In this context, ψ describes the dimensionless field amplitude, while x and t denote transversal and longitudinal coordinates and the above problem transforms into the search of a periodic modulation of the nonlinearity assuring the strong suppression of the cross-phase modulation of the beams bordering edges of the same band gap.

Let us consider a linear superposition of two solutions $\psi_e=\phi_e(x)e^{-i\mu_e t}$ and $\psi_o=\phi_o(x)e^{-i\mu_o t}$ of eq. (1) with $\mathcal{F}(t)\equiv 0$, of the form

$$\psi(x,t)=\psi_e(x,t)+e^{i\Theta}\psi_o(x,t). \quad (2)$$

Here $\phi_e(x)=\phi_e(-x)$ and $\phi_o(x)=-\phi_o(-x)$ denote even and odd solutions of the stationary GPE:

$$\mu_{e,o}\phi_{e,o}=-\frac{\mathrm{d}^2\phi_{e,o}}{\mathrm{d}x^2}-V\cos(2x)\phi_{e,o}+\mathcal{G}(x)|\phi_{e,o}|^2\phi_{e,o}, \quad (3)$$

with corresponding chemical potentials, $\mu_{e,o}$, bordering opposite edges of the same band gap. The existence of such solutions can be assured by a proper design of $\mathcal{G}(x)$ [9,12], assumed in the following to be an even function of x: $\mathcal{G}(x)=\mathcal{G}(-x)$. Since $\phi_{e,o}(x)$ have constant phases, they can be taken to be real, with the relative phase Θ in (2) accounting for possible phase mismatches. Having opposite symmetries, the nonlinear states $\phi_{e,o}(x)$ are automatically orthogonal $\int\phi_e(x)\phi_o(x)\mathrm{d}x=0$. The Hamiltonian of eq. (1) with $\mathcal{F}(t)\equiv 0$

$$H=\int_{-\infty}^{\infty}\left[\left|\frac{\partial\psi}{\partial x}\right|^2-V\cos(2x)\,|\psi|^2+\frac{\mathcal{G}(x)}{2}\,|\psi|^4\right]\mathrm{d}x \quad (4)$$

for the superposition state (2) can be written as $H=H_e+H_o+H_{int}(t)$, with terms $H_{e,o}$ corresponding to the Hamiltonian of solutions $\psi_{e,o}(x,t)$:

$$H_\alpha=\int_{-\infty}^{\infty}\left[\left(\frac{\mathrm{d}\phi_\alpha}{\mathrm{d}x}\right)^2-V\cos(2x)\phi_\alpha^2+\frac{\mathcal{G}(x)}{2}\phi_\alpha^4\right]\mathrm{d}x,$$

(with the subscript α standing for e,o) and with

$$H_{int}(t)=\left\{2\cos^2\left[(\mu_e-\mu_o)t-\Theta\right]+1\right\}\times\int_{-\infty}^{\infty}\mathcal{G}(x)\phi_e^2(x)\phi_o^2(x)\mathrm{d}x \quad (5)$$

characterizing the nonlinear interaction between the solutions. The superposition state (2) is not a solution of the GPE, what is manifested in the Hamiltonian with the presence of the interaction term $H_{int}(t)$. One can make the superposition state to be a remarkably good approximation to an exact solution by taking the nonlinearity function $\mathcal{G}(x)$ so to kill the interaction term (5) and satisfying at the same time the requirement for the effective nonlinearities at the gap edges to have opposite signs. This amounts to taking $\mathcal{G}(x)$ that satisfies the conditions

$$\int_{-\infty}^{\infty}\mathcal{G}(x)\phi_e^2(x)\phi_o^2(x)\mathrm{d}x=0, \quad (6a)$$

$$\left[\int_{-\infty}^{\infty}\mathcal{G}(x)\phi_e^4(x)\mathrm{d}x\right]\times\left[\int_{-\infty}^{\infty}\mathcal{G}(x)\phi_o^4(x)\mathrm{d}x\right]<0. \quad (6b)$$

A remarkable fact is that these conditions can be satisfied even with a simple cos-like spatial dependence of the nonlinearity. In the following we take $\mathcal{G}(x)$ to be of the form $\mathcal{G}(x)=\sigma+G_1\cos(2x)$, with $\sigma=\pm 1$.

We remark that besides cos-like potentials our approach can be used for any potential leading to two-level systems with controllable interactions, such as vector-like or multi-component NLS solitons with spatial sign-varying nonlinearity. The choice of OLs here resides in the fact that GSs are well established objects both in BEC and in nonlinear optics and techniques for their manipulations are well developed so that experimental implementations of the superposition state become feasible (see below).

By properly choosing the amplitude G_1 (see fig. 1(a)) one can provide the condition $H_{int}(t)\equiv 0$ for a wide range of chemical potentials μ_e, μ_o. A particular example of even and odd GSs with chemical potentials belonging to the lowest band gap, is illustrated in fig. 1(a). We remark that not all the possible GSs are dynamically stable. Specific examples of stable GSs are shown in figs. 1(b)–(e). The predicted stability of the linear superposition of the

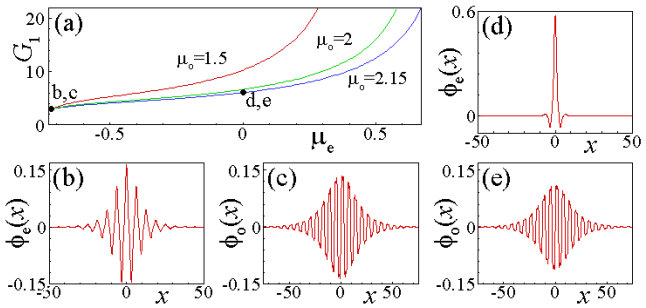

Fig. 1: (Color online) (a) Amplitude of the nonlinear lattice G_1 *vs.* chemical potential μ_e of an even GS which ensures the conditions (6) for $V=3$, and $\sigma=-1$. Panels (b)–(e) show pairs of stable GSs at the corresponding points of panel (a) for $\mu_o=2.15$. In (b) and (c) even and odd GSs at $G_1=3.0373$ with the parameters $\mu_e=-0.72$, $N_e=0.13$ and $N_o=0.32$. In (d) and (e) even and odd GSs at $G_1=6.0623$ with the parameters $\mu_e=0$, $N_e=0.51$, and $N_o=0.213$. The first lowest gap is $\mu\in(-0.7332, 2.1651)$. The quantities $N_{o,e}$ are defined in (7).

GSs, shown in figs. 1(b), (c), is confirmed by numerical integration of GPE (1) with $\mathcal{F}(t)=0$.

To this end we used in all numerical simulations perturbed initial conditions of the form: $(1+0.1\cos(21x))\varphi$ with φ denoting the unperturbed superposition solution. Stability was also checked by using unperturbed initial conditions but with a noisy modulations of the nonlinearity (see fig. 4). The resulting projections of the wave function $\psi(x,t)$ on the stationary solutions, *i.e.*

$$c_{e,o}(t)=\frac{1}{N_{e,o}}\int_{-\infty}^{\infty}\psi(x,t)\phi_{e,o}(x)\mathrm{d}x, \tag{7}$$

where $N_{e,o}=\int_{-\infty}^{\infty}\phi_{e,o}^2(x)\mathrm{d}x$ are numbers of atoms in the even and odd GSs, are depicted in fig. 2(a) (notice that $c_{e,o}(t)$ describe numbers of atoms in the GS state, rather than total populations of the respective energy levels, and as such they are only normalized by the initial conditions $c_e(0)=c_o(0)=1$). One observes that the square modula of the projections are approximately equal to unity (the difference is less than 0.2%) during the relatively long integration time.

From numerical simulations it follows that for small-amplitude GSs the stability of the superposition is less sensitive to variations of the amplitude G_1. In particular, when the nonlinearity amplitude is bigger than the optimal one, the superposition state can survive even under relatively large G_1 (see fig. 2(c)). For nonlinearity amplitudes below the optimal value the superposition state is destroyed only after the nonlinearity is reduced by a factor $\sim 2/3$ of its optimal value (see fig. 2(b) and inset). The stability of linear superpositions of large-amplitude GSs, however, is much more sensitive to nonlinearity amplitude changes (cf. figs. 2(c) and (f)). In particular from fig. 2(d) we see that while the linear superposition of a large-amplitude even GS and a small-amplitude odd GS

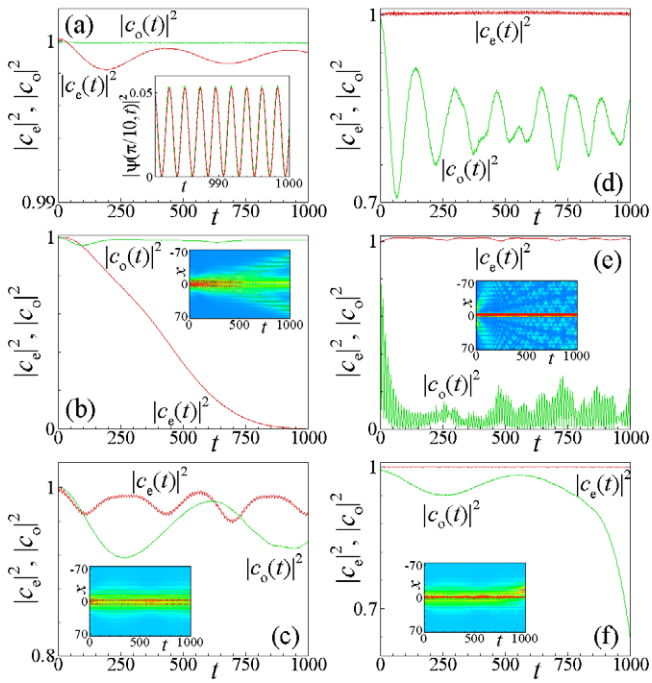

Fig. 2: (Color online) Time evolution of the projections $|c_e|^2$ and $|c_o|^2$ for $V=3, \sigma=-1$, and $G_1=3.0373$ (a), $G_1=2.0$ (b), $G_1=10.0$ (c,f), $G_1=6.0623$ (d), $G_1=2.5$ (e). The initial conditions are taken as in (2) with $\Theta=0$ and with the chemical potentials of the odd and even GSs given by: $\mu_o=2.15$, $\mu_e=-0.72$ (a)–(c), $\mu_e=0$ (d)–(f). The inset in (a) shows the time evolution of the particle density $|\psi(x,t)|^2$ at $x=\pi/10$ as obtained from the numerical integration of the GPE (solid lines, parameters are the same as in panel (a)), and from eq. (8) (dashed lines). Insets in panels (b), (c) and (e), (f) show the destruction of the superposition in the coordinate space.

(depicted in figs. 1(d), (e)) can be stable, for an improper choice of the nonlinearity amplitude the superposition state is quickly destroyed (figs. 2(e), (f) and insets).

These different behaviors can be explained by observing that for $G_1\gg 1$ one has that $\mathcal{G}(x)\approx G_1\cos(2x)$ so that the condition eq. (6a) is practically always satisfied for small-amplitude GSs because, being such solutions very extended, the function $|\phi_e(x)\phi_o(x)|^2$ is effectively averaged to zero after multiplication by $\mathcal{G}(x)$. This is not the case for big-amplitude GSs whose linear superpositions are much more sensitive to the nonlinearity amplitude variations because, being the solutions more localized, the averaging out is less effective. Notice from eq. (2) that the total density $\rho(x,t)=|\psi(x,t)|^2$ can be written in terms of the stationary densities $\rho_{e,o}(x)=\phi_{e,o}^2(x)$ as

$$\rho(x,t)=\rho_o(x)+\rho_e(x)+2\phi_o(x)\phi_e(x)\cos[(\mu_o-\mu_e)t+\Theta] \tag{8}$$

with the time-dependent term in the right-hand side representing the contribution due to the interference. Also notice that the interference term disappears both if averaged on time (denoted by $\langle\cdot\rangle$ in the following) or integrated on space, this giving $\langle\rho\rangle=\rho_o+\rho_e$ and

$N = N_o + N_e$, respectively. This has to be complemented with the energy conservation written in the form $\langle H \rangle = \langle H_o \rangle + \langle H_e \rangle$.

In the inset in fig. 2(a) we have compared the time evolution governed by eq. (8) with the one obtained directly from the integration of the GPE (1) with $\mathcal{F}(t) = 0$. One can clearly see that even on a long time scale the discrepancy between the numerical density profile of the superposition-state and the one obtained from eq. (8) is remarkably small. In order to observe the density fluctuations induced by the interference term one should be able to measure variations of the density profile of the superposition state on a time scale of the order of the period of the oscillation $T = 2\pi/(\mu_o - \mu_e)$. For the parameters used in figs. 2(a) and (c) this time is $T \approx 2$ and $T \approx 3$, correspondingly. This time, however, can be made longer by suitably designing the linear OL to reduce the width of the first gap.

Let us now discuss how the superposition of GSs could be created in a BEC experiment. To this regard we remark that GSs with negative effective masses (repulsive interactions) at the edge of the Brillouin zone have already been created [20]. For repulsive interactions the GSs can exist only near the top edges of the bands. On the other hand, it has been theoretically demonstrated in [7] that a proper periodic modulation of the scattering length in space makes it possible to have stable GSs near both top and bottom edges of the first and second band, respectively. A GS near the bottom edge of the second band can be produced by transferring a first band GS to the second band by means of an half-cycle of Rabi-oscillation as discussed in [9]. This implies that by starting from two spatially separated GSs of the same band and by tunneling one of them to the second band, two spatially separated stationary GSs with chemical potentials located near the opposite edges of the first band gap can be created. These states will obviously have opposite effective masses and opposite spatial symmetries (one odd and the other even) and by construction they result to be orthogonal. After this preparation, the next problem is to put the states into a linear superposition. This can be done by observing that by knowing the positions of the even and odd GSs in the OL one can design a time-dependent force (*e.g.* lattice accelerations) to bring them in the same intermediate position so to form the macroscopic superposition. In this respect we recall that the effect of the external force $\mathcal{F}(t)$ on the GS dynamics is well described by the semiclassical equation of motion [7,8]: $\mathrm{d}x/\mathrm{d}t = v(q) \equiv \mathrm{d}E(q)/\mathrm{d}q$, $\mathrm{d}q/\mathrm{d}t = -\mathcal{F}(t)$, where x and q denote the GS centers in real and in reciprocal spaces, respectively, $v(q)$ is the Bloch velocity. To create the superposition state one can use a time-dependent force of the form

$$\mathcal{F}(t) = \begin{cases} \gamma, & t < t_1, \\ 0, & t_1 \leqslant t < t_2 \quad \text{and} \quad t \geqslant t_2 + t_1, \\ -\gamma, & t_2 \leqslant t < t_2 + t_1. \end{cases} \tag{9}$$

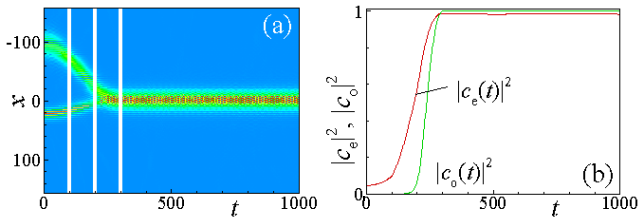

Fig. 3: (Color online) (a) Spatio-temporal evolution of the even and odd GSs initially located at $x = 7\pi$ and $x = -30\pi$, respectively, under the action of the external force (9) with $\gamma = -0.001$, and times $t_1 = 100$, $t_2 = 200$, $t_{stop} = 300$ which are indicated by the vertical lines. (b) Time evolution of the projections on the initial states during the superposition formation. The parameters of the solitons, OL, and nonlinearity are the same as in fig. 2(a).

From the equation of motion it follows that the two spatially separated GSs, during the time t_1 will accelerate towards each other, due to their opposite effective masses. In the time interval $t_1 \leqslant t < t_2$, when the external force is switched off, the even and odd GSs will move with constant velocities given by $v_e = \mathrm{d}E_1(q = 1 - \gamma t_1)/\mathrm{d}q$ and $v_o = \mathrm{d}E_2(q = 1 - \gamma t_1)/\mathrm{d}q$, respectively, while in the interval $t_2 \leqslant t < t_2 + t_1$, their velocities gradually decrease to zero (due to equality of absolute values of external forces at accelerating and decelerating intervals and equal durations of these intervals) and at the time $t_{stop} = t_2 + t_1$ the solitons will become stationary. If the initial distance between the GSs is

$$\begin{aligned} \Delta x \;=\; & 2[E_2(1 - \gamma t_1) - E_1(1 - \gamma t_1) - E_2(1) + E_1(1)] \\ & + (|v_e| + |v_o|)(t_2 - t_1), \end{aligned}$$

at $t = t_{stop}$ their centers will coincide, and by switching off the force from this time on, a stable linear superposition of the two GSs will be formed.

The described process is illustrated in fig. 3(a) for solitons initially separated by a distance $\Delta x = 37\pi$. Notice that once the superposition is formed it remains stable for long time, without any apparent loss of matter and with the number of atoms preserving the relation $N = N_e + N_o$ in time. This is also evident from fig. 3(b) where the projections of the state along the even and odd components are depicted during the time process of state formation and after. We see that the projections rapidly approach the unity as the states become closer and practically coincide with 1 when the superposition is formed. The possibility to create superpositions in a noisy environment and the structural stability of the superposition state has been ensured by adding a uniformly distributed random perturbation of zero average to the nonlinearity function. From figs. 4(a) and (b) we see that the superposition can still be formed in the presence of noise. We find that at time $t_{stop} = 300$, when external force is switched off and superposition state is formed, projections deviate from unity by less than 1.5% in fig. 4(a) and less than 3%

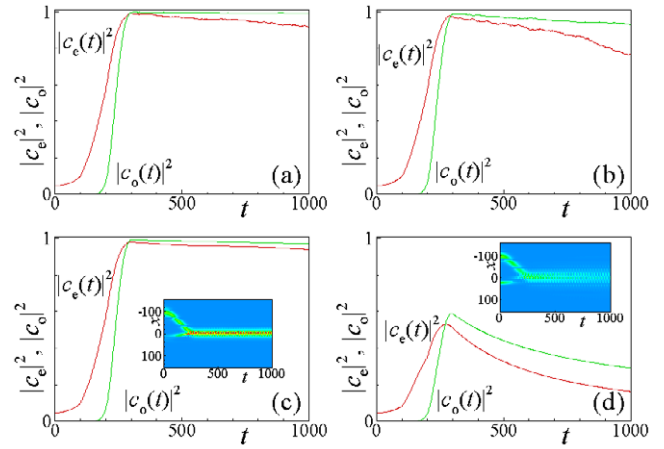

Fig. 4: (Color online) (a,b) The same as in fig. 3(b) but with a small random perturbation added to the nonlinearity $G(x) = -1 + 3.0373\cos(2x) + aR(x,t)$, where $a = 5$ (a) or $a = 10$ (b) is the amplitude of random noise, $R(x,t)$ is the noise function, uniformly distributed in the interval $[-1, 1]$. (c), (d) The same as in fig. 3(b) but with a nonlinear dissipation added to the nonlinearity $G(x) = -1 + 3.0373\cos(2x) - i\nu$, where $\nu = 0.001$ (c) or $\nu = 0.1$ (d) is the amplitude of dissipation. Insets in panels (c), (d) show the spatio-temporal evolution of soliton superposition in the dissipative BEC.

in fig. 4(b). Notice that in spite of the big amplitude of the random noise, the linear superpositions destroys very slowly with time.

An interesting question to ask is whether the superposition state can be formed also in the presence of a small dissipation. In this respect, we remark that a small imaginary part of the scattering length can arise due to spontaneous emission losses inherent to the optically induced Feshbach resonance technique [21,22]. Such dissipative effects can be strongly reduced by using laser fields with sufficiently high intensity and detuned from the resonance. In particular, it has been shown that by using a laser light near the resonance with a molecular bound-to-bound transition in ^{87}Rb BEC it is possible to shift the value of the magnetic field where the Feshbach resonance occurs and to vary the scattering length on the optical wavelength scale with a considerable loss reduction (about two orders of magnitude lower than usual Feshbach experiments) [23].

The effect of a small dissipation on the superposition state has been investigated by adding a small imaginary component to the nonlinearity function in the GPE, *i.e.* by taking $\mathcal{G}(x) = \sigma + G_1\cos(2x) + i\nu$, with $\sigma = \pm 1$ and $\nu \ll 1$. From panels (c), (d) of fig. 4 we see that in this case the superposition state can still be formed (although in non-optimal manner since at formation time $t = 300$ projections $|c_{e,o}|^2$ slightly deviate from unity) and, although slowly decaying (due to dissipation), it does not lose its internal structure, *i.e.* the state remains in the superposition during decay (cf. insets in figs. 4(c), (d) with those in figs. 2(b), (e) where the superposition is destroyed). For a small dissipation the decay occurs on a time scale which is long enough for the superposition state to be observed.

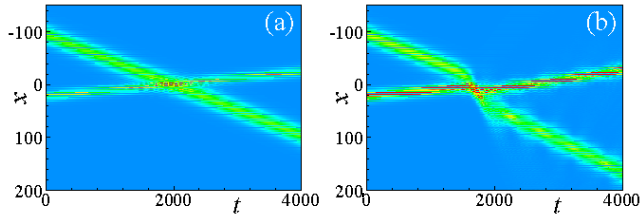

Fig. 5: (Color online) Spatio-temporal evolution of the even and odd GS states initially located at $x = 7\pi$ and $x = -30\pi$, respectively, under the action of the external force $\mathcal{F}(t) = -0.001$, which is switched off at $t = 10$. The parameters of solitons and the nonlinearity are the same as in respective panels of fig. 2.

The possibility to control the soliton-soliton interaction by means of spatial periodic modulations of the nonlinearity is also demonstrated in fig. 5 by means of scattering processes of two GSs with chemical potential bordering opposite band gap edges. Figure 5(a) refers to the case of an optimal design of the nonlinearity (*i.e.* the soliton-soliton interaction is fully suppressed) while in fig. 5(b) we show the case of scattering for a non-optimal design of the nonlinearity. Opposite to fig. 5(b) (and to usual scattering of solitons of integrable models), in fig. 5(a) no phase shift of the scattered trajectories is visible, this being a consequence of the total absence of interaction. In the case of a non-optimal design, the interaction between the solitons is evident from the strong perturbation of the trajectories even for small interaction times (after the interaction the velocities are practically unchanged but there is a visible shift of the trajectories which is larger for the GS with the smaller effective mass).

The obtained results can be generalized to other nonlinear solutions of the GPE. As an example we consider the case of periodic matter waves whose chemical potentials belong to the first (the periodic waves are even) or to the second (the periodic waves are odd) bands of the OL spectrum. Similarly to the GS case, the condition $H_{int}(t) \equiv 0$ can be achieved also for periodic waves with the only difference that now H_{int} is given by eq. (5) with the integration over the whole real axis is replaced by the integration over the unit cell $[-\pi/2, \pi/2]$ and the number of particles per unit cell are given by $N_{e,o} = \int_{-\pi/2}^{\pi/2} \phi_{e,o}^2(x)\mathrm{d}x$. An example is shown in fig. 6(a), where we considered a repulsive averaged nonlinearity ($\sigma = 1$) and a π-shifted nonlinearity modulations (expressed by a negative amplitude: $G_1 < 0$). Similarly to the GS case, among odd periodic waves only low-amplitude ones are stable, corresponding in fig. 6(a) to the branch $\mu_o = 2.2$ only. The possibility to create stable superposition state of periodic waves, shown in figs. 6(b) and (c), is confirmed by fig. 6(d). Similar results are expected to apply also to dark solitons and multi-soliton solutions.

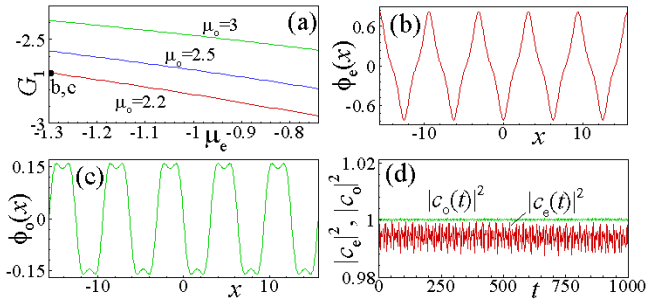

Fig. 6: (Color online) (a) Amplitude of the nonlinearity G_1 *vs.* even periodic wave chemical potential μ_e, which provides the conditions $H_{int} \equiv 0$ in the case of OL amplitude $V = 3$, nonlinearity average value $\sigma = 1$ and different odd periodic wave chemical potentials. (b), (c) Periodic wave shapes at corresponding points of panel (a) for parameters $\mu_e = -1.3$, $G_1 = -2.7$, $N_e = 0.74$ (b); $\mu_o = 2.2$, $G_1 = -2.7$, $N_o = 0.056$ (c). (d) Time evolution of the projections $|c_e|^2$, $|c_o|^2$, obtained from the numerical integration of GPE (1) with $\mathcal{F}(t) = 0$ and with initial condition taken as the linear superposition (2) with $\Theta = 0$, of the periodic waves depicted in panels (b), (c).

An estimate of the parameters for a BEC experimental observation of the superposition can be made by referring to a ^{7}Li condensate in a trap with $a_\perp = 2\,\mu$m, $d = 1\,\mu$m and with $\langle a_s \rangle = -2$ nm, created by an optically induced Feshbach resonance. The linear superposition depicted in fig. 2(a) can be achieved by putting $\approx$200 atoms (this corresponding to $N_e = 0.13$) in the even soliton and $\approx$500 atoms (this corresponding to $N_o = 0.32$) in the odd soliton, provided the amplitude of modulation of the scattering length is $\sim$6.075 nm (corresponding to $G_1 = 3.0373$). The way of creation of the superposition state, presented in fig. 3, can be realized during $\sim$6.75 ms (this corresponds to $t_{stop} = 300$), using an external force $1.474 \cdot 10^{-26}$ N (for which $\gamma = 0.001$) applied to the pair of solitons, initially separated by 37 μm.

In conclusion we have shown that it is possible to control the interaction between two stationary solution characterized by opposite parity of the Gross-Pitaevskii equation (or in more general contexts by the nonlinear Schrödinger equation with periodic coefficients) by means of spatially dependent nonlinear interactions. We showed that this permits to eliminate (or strongly suppress) the interaction (cross-phase modulation in an optical context) between GSs bordering opposite edges of a band gap, and to create stable linear superpositions of them which survive for very long times. This results in the effective decoupling of the dynamics of the two states and offers the possibility to efficiently manipulate them as linear objects. We showed that superposition states are remarkably robust against perturbations and can exist even in the presence of a small dissipation, a fact which should make them be observed in real experiments, both with matter waves and with light propagation in nonlinear optical waveguide arrays.

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