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Irreducible many-body Casimir energies of intersecting objects

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Abstract – The vacuum energy of bosonic fields interacting locally with objects is decomposed into irreducible many-body parts. The irreducible N-body contribution is finite if the N objects have no common intersection, $O_1 \cap O_2 \cdots \cap O_N = \emptyset$. The perturbative high-temperature expansion of the corresponding irreducible N-body spectral function $\tilde{\phi}^{(N)}(\beta \sim 0)$ vanishes to all orders even if some of the objects overlap. These irreducible spectral functions and their associated Casimir energies can, in principle, be computed numerically or approximated semi-classically without regularization or implicit knowledge of the spectrum. They are analytic in the parameters describing the relative orientation and position of the individual objects and remain finite when some, but not all, of the N objects overlap. The finiteness of the irreducible N-body Casimir energy of a massless scalar field with potential scattering is explicitly verified and found to be negative for an even, and positive for an odd number of objects. The sign in this case does not depend on the strength of the local N-body potentials. Some simple examples illustrate the analyticity of N-body Casimir energies. A multiple scattering representation of the irreducible three-body Casimir energy is derived. It remains finite when the three objects overlap only pairwise.

Introduction. – The Casimir energy for two disjoint bodies is finite and may be estimated [1–5]. It can, in principle, be computed to arbitrary numerical precision [6–8]. For disjoint bodies, the multiple scattering representation of the interaction energy [9–12] thus solves many problems encountered in technological applications [6,13]. We here develop an extension of this formalism and extract finite irreducible Casimir energies for more than two bodies that are not necessarily mutually disjoint. The analysis gives a new interpretation to finite parts of zero-point energies that could provide a framework for exploring gravitational effects due to vacuum energies [14] and result in a more systematic approach to Casimir self-stresses for arbitrarily shaped bodies.

For clarity of presentation and to avoid infrared issues, we assume that the objects $\{O_i, i=1,\ldots,N\}$ are all embedded in a large, but finite, connected Euclidean region $\mathfrak{D}_\emptyset$ of dimension d. The thermodynamic limit $\mathfrak{D}_\emptyset \to \mathcal{R}^d$ may be taken at the end of the calculation. Formally, the vacuum energy $\mathcal{E}_{12\ldots N}$ of a massless bosonic field in the presence of N objects may be decomposed as

$$\mathcal{E}_{12\ldots N} = \mathcal{E}_\emptyset + \sum_i \tilde{\mathcal{E}}_i^{(1)} + \sum_{i<j} \tilde{\mathcal{E}}_{ij}^{(2)} + \cdots + \tilde{\mathcal{E}}_{12\ldots N}^{(N)}, \qquad (1)$$

where $\tilde{\mathcal{E}}_{i_1\ldots i_k}^{(k)}$ is the irreducible contribution to the vacuum energy that depends on k objects $O_{i_1}\ldots O_{i_k}$ only. Equation (1) recursively defines the irreducible N-body Casimir energy $\tilde{\mathcal{E}}_{12\ldots N}^{(N)}$ as an alternating sum of vacuum energies (see eq. (12)). The irreducible two-body Casimir energy $\tilde{\mathcal{E}}_{12}^{(2)}$ for instance is

$$\tilde{\mathcal{E}}_{12}^{(2)} = \mathcal{E}_{12} - \mathcal{E}_1 - \mathcal{E}_2 + \mathcal{E}_\emptyset. \qquad (2)$$

It is finite [11] for two disjoint objects. Below I show that the irreducible N-body Casimir energy $\tilde{\mathcal{E}}_{12\ldots N}^{(N)}$ is finite as long as the N objects have no *common* intersection, $O_1 \cap O_2 \cdots \cap O_N = \emptyset$. For two objects this requires them to be disjoint, but three and more objects need not be mutually disjoint. The irreducible three-body contribution to the Casimir energy of a triangle, for instance, is finite. For a massless scalar field whose interaction with objects is modeled by positive *local* potentials, irreducible N-body Casimir energies are found to be negative for an even, and positive for an odd number of objects.

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Subtracted N-body spectral functions. – Some irreducible vacuum energies, such as self-energies $\tilde{\mathcal{E}}^{(1)}$, may diverge when the associated objects overlap. Equation (1) is formal and requires a high-frequency cutoff to be meaningful. However, spectral functions generally are finite and well defined even when one-loop vacuum energies are not. They can be similarly decomposed into irreducible parts and we therefore relate the irreducible N-body Casimir energy $\tilde{\mathcal{E}}^{(N)}$ to the corresponding irreducible N-body spectral function $\tilde{\phi}^{(N)}(\beta)$,

$$\tilde{\mathcal{E}}^{(N)} = -\frac{\hbar c}{\sqrt{8\pi}} \int_0^\infty \tilde{\phi}^{(N)}(\beta) \frac{\mathrm{d}\beta}{\beta^{3/2}}. \tag{3}$$

$\tilde{\phi}^{(N)}(\beta)$ is constructed as follows. Let $\mathfrak{D}_s$ represent the domain $\mathfrak{D}_\emptyset$ with objects $\{O_j; j \in s\}$ embedded, $\mathfrak{D}_{1\ldots N}$ being the finite domain $\mathfrak{D}_\emptyset$ with all N objects included. Denote with $\mathfrak{P}(s)$ the power set of the elements of a set s of finite cardinality $|s| \leqslant N$ and let $\mathfrak{P}_N = \mathfrak{P}(\{1\ldots N\})$. Let $\phi_s(\beta)$ be the spectral function, or trace of the heat kernel $\mathfrak{K}_{\mathfrak{D}_s}$, for the domain $\mathfrak{D}_s$,

$$\phi_s(\beta) = \mathrm{Tr}\, \mathfrak{K}_{\mathfrak{D}_s}(\beta) = \sum_{n\in\mathbb{N}} e^{-\beta\lambda_n(\mathfrak{D}_s)/2}. \tag{4}$$

Here $\{\lambda_n(\mathfrak{D}_s) > 0, n \in \mathbb{N}\}$ is the spectrum of a bosonic field that vanishes on the boundary of $\mathfrak{D}_\emptyset$ and whose interactions with the objects in $\mathfrak{D}_s$ are *local*. We assume the interaction of the field with the objects is described by (compatible) local boundary conditions or by positive local potentials. The considerations of this section are not restricted to scalar fields and apply to any local bosonic field theory with cluster decomposition property, including quantum electrodynamics without free charges.

The irreducible spectral function $\tilde{\phi}^{(N)}(\beta)$ in eq. (3) is given by the following alternating sum of spectral functions $\phi_s(\beta)$ for the individual domains D_s:

$$\tilde{\phi}^{(N)}(\beta) := \sum_{s\in\mathfrak{P}_N} (-1)^{N-|s|} \phi_s(\beta). \tag{5}$$

This is a special case of the geometrical subtraction procedure advocated in ref. [15]. We will see that the asymptotic power expansion of $\tilde{\phi}^{(N)}(\beta)$, defined in eq. (5), for $\beta \sim 0$ vanishes to all orders if the *common* intersection of all N objects is empty. A pictorial representation of eq. (5) for four line segments in a bounded two-dimensional Euclidean space is shown in fig. 1.

To facilitate the proof that the integral in eq. (3) is finite, we demand that individual heat kernels are uniformly bounded by the free heat kernel of $\mathcal{R}^d$,

$$0 < \mathfrak{K}_{\mathfrak{D}_s}(\mathbf{x}, \mathbf{y}; \beta) \leqslant K(2\pi\beta)^{-d/2} e^{-(\mathbf{x}-\mathbf{y})^2/(2\beta)}, \tag{6}$$

for some finite $K > 0$. For a scalar interacting with local positive potentials this is implied by the Feynman-Kac theorem [16]: the heat kernel is the transition probability for Brownian motion and a (positive) potential *reduces* it. Dirichlet boundary conditions, in particular, may be imposed on a surface by killing any path that crosses. Because any condition will only be satisfied by a reduced number of paths, the bound of eq. (6) may also hold for objects represented by other types of local boundary conditions. The following considerations require only that correlation functions vanish faster than any power of β as $\beta \to 0$ for any *finite* distance $|\mathbf{x} - \mathbf{y}| > \delta > 0$. One thus may be able to relax the uniform bound of eq. (6) considerably.

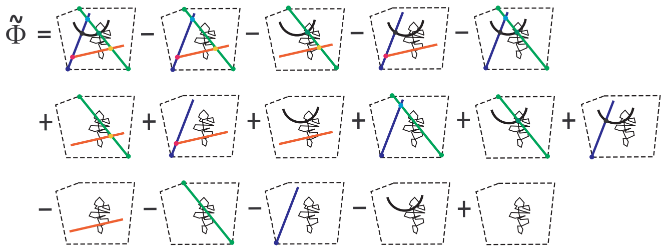

Fig. 1: (Colour on-line) The subtracted spectral function $\tilde{\phi}^{(N)}(\beta)$ defined in eq. (5) for a bounded two-dimensional domain $\mathfrak{D}_\emptyset$ with four intersecting line segments as objects. Each pictograph represents the spectral function of the corresponding domain taken with the indicated sign. Various local features that contribute to the asymptotic expansion of each spectral function at high temperatures (small β) have been highlighted: lines of different color correspond to possibly different, but compatible, boundary conditions or local potentials. Intersections of line segments that differ locally are shown in different colors. The contribution to the asymptotic expansion due to any particular local feature vanishes: the total signed number of times any particular line segment contributes is zero, as is the total signed number of times any particular vertex occurs. A random walk that crosses only three of the four segments is shown schematically. It does not contribute to this irreducible spectral function of a massless scalar field in the Feynman-Kac path integral.

$\phi_s(\beta)$ can be interpreted as a bosonic single-particle partition function at inverse temperature β and a positive semi-definite spectrum is equivalent to the absence of tachyons in the causal local theory. The spectral functions $\phi_s(\beta)$ of eq. (5) in this case are positive and monotonically decreasing with β.

For local bosonic field theories, the asymptotic expansion of $\phi_s(\beta)$ at small β has the general form [17–20],

$$\phi_s(\beta \sim 0) \sim \sum_{\nu=-d}^{\infty} (2\pi\beta)^{\nu/2} A_s^{(\nu)} + \mathcal{O}(e^{-\ell_{\mathrm{min}}^2/(2\beta)}), \tag{7}$$

where the Hadamard-Minakshisundaram-DeWitt-Seeley coefficients $A_s^{(\nu)}$ for the domain $\mathfrak{D}_s$ have canonical length-dimension $(-\nu)$. Note that if eq. (6) holds, exponentially suppressed terms are associated with classical periodic paths of finite length ℓ_{min}. We decompose the heat kernel coefficients $A_s^{(\nu)}$ of eq. (7) into parts arising from local features of the individual objects and their overlaps,

$$A_s^{(\nu)} = \sum_{\tau\in\mathfrak{P}(s)} a_\tau^{(\nu)}, \tag{8}$$

where the sum extends over all ($|s|!$) sets in the power set $\mathfrak{P}(s)$ of the set s. Equation (8) defines reduced heat kernel coefficients $a_\tau^{(\nu)}$ recursively: the $a_\emptyset^{(\nu)}=A_\emptyset^{(\nu)}$ are the heat kernel coefficients associated with the Euclidean domain $\mathfrak{D}_\emptyset$; the $a_{\{j\}}^{(\nu)}$ give their *change* when object j is inserted; the $a_{\{jk\}}^{(\nu)}$ account for further changes in the heat kernel coefficients due to local overlaps of objects j and k. Note that the $a_\tau^{(\nu)}$ are *not* the heat kernel coefficients of the domain $\mathfrak{D}_\tau$ —they are their irreducible part only and arise from arbitrary short correlations near *common* intersections of the objects in the set τ. $a_{\{jk\}}^{(\nu)}=0$ for two disjoint objects j and k, if we assume (as implied by eq. (6)) that asymptotic correlations over finite distances $|\mathbf{x}-\mathbf{y}|>\delta>0$ vanish faster than any power in β. Similarly, $a_{\{123\}}^{(\nu)}=0$ if $O_1\cap O_2\cap O_3=\emptyset$. $a_{\{123\}}^{(\nu)}=0$ vanishes even if the three objects are not pairwise disjoint, since $a_{\{jk\}}^{(\nu)}$ already accounts for contributions due to the intersection $O_j\cap O_k$.

This argument can be extended to N objects. For local interactions the irreducible correction,

$$a_{\{1\dots N\}}^{(\nu)}=0, \quad \text{if} \quad O_1\cap\cdots\cap O_N=\emptyset. \tag{9}$$

Note again that the condition in eq. (9) does not imply that the objects have to be mutually disjoint (except for $N=2$). It now is a combinatoric matter to show that for $\tau\subsetneq\{12\dots N\}$ the contribution of any $a_\tau^{(\nu)}$ to the asymptotic expansion of $\tilde{\phi}^{(N)}(\beta)$ in eq. (5) vanishes. Because the other $|s|-|\tau|$ objects may be selected from the remaining $N-|\tau|$ in any order, the number of times the set τ occurs as a subset of the sets in $\mathfrak{P}_N$ (for $|s|\geqslant|\tau|$) is the combination $\frac{(N-|\tau|)!}{(N-|s|)!(|s|-|\tau|)!}=\binom{N-|\tau|!}{N-|s|!}$. For $N>|\tau|$ the contribution to the asymptotic expansion of $\tilde{\phi}^{(N)}(\beta)$ in eq. (5) proportional to $a_\tau^{(\nu)}$ then is

$$(2\pi\beta)^{\nu/2}a_\tau^{(\nu)}\sum_{|s|=|\tau|}^{N}(-1)^{N-|s|}\binom{N-|\tau|}{N-|s|}=0. \tag{10}$$

When N objects have no *common* intersection, the asymptotic expansion of the irreducible N-body spectral function $\tilde{\phi}^{(N)}(\beta)$ thus has the form,

$$\tilde{\phi}^{(N)}(\beta\sim 0)\sim\mathcal{O}(e^{-\ell_{\min}^2/(2\beta)}), \tag{11}$$

and vanishes faster than any power of β. This may be explicitly verified in examples like the one shown in fig. 1, noting that contributions to the asymptotic expansion proportional to the volume, surfaces, corners, curvatures etc..., all cancel. Together with the fact that the spectral functions $\phi_s(\beta)$ decay monotonically and remain bounded for large β, the asymptotic behavior of eq. (11) implies that the irreducible Casimir energy in eq. (3) is *finite* if the objects have no *common* intersection.

The subtraction procedure allows one to formally interpret $\tilde{\mathcal{E}}^{(N)}$ as an alternating sum of vacuum energies $\mathcal{E}_s$ associated with the domains $\mathfrak{D}_s$,

$$\tilde{\mathcal{E}}^{(N)}=\sum_{s\in\mathfrak{P}_N}(-1)^{N-|s|}\mathcal{E}_s. \tag{12}$$

The sum on the right side of eq. (12) requires some regularization to be meaningful but, if this procedure does not explicitly depend on the specific domain $\mathfrak{D}_s$ (for instance proper time regularization), the previous considerations show that the irreducible N-body contribution $\tilde{\mathcal{E}}^{(N)}$ remains well defined as the regularization is removed. The absence of a power series in the asymptotic expansion of $\tilde{\phi}^{(N)}(\beta\sim 0)$ also explains why a semi-classical approach based on classical periodic orbits tends to approximate Casimir energies fairly well [1–3,21–23]: it reproduces the leading exponentially suppressed terms of the asymptotic expansion.

Massless scalar field with local potential interactions. – This subtraction procedure is particularly transparent for a massless scalar field in a bounded Euclidean space $\mathfrak{D}_\emptyset$ whose interaction with N objects is described by a local (positive) potential $V=\sum_{i=1}^N V_i$. The path integral in this case not only provides for an alternate proof of the finiteness of the irreducible N-body contribution but also determines its sign. Using the world-line approach of [7] for potential scattering, the Feynman-Kac theorem [16] generally states that

$$\phi_s(\beta)=\int_{\mathfrak{D}_\emptyset}\frac{\mathrm{d}\mathbf{x}}{(2\pi\beta)^{d/2}}\mathcal{P}_{\mathfrak{D}_s}[\ell_\beta(\mathbf{x})], \tag{13}$$

where $\mathcal{P}_{\mathfrak{D}_s}[\ell_\beta(\mathbf{x})]$ is the probability for a standard Brownian bridge[1], $\ell_\beta(\mathbf{x})=\{\mathbf{x}_t,\ 0\leqslant t\leqslant\beta;\ \mathbf{x}_0=\mathbf{x}_\beta=\mathbf{x}\}$, that starts at $\mathbf{x}$ and returns to $\mathbf{x}$ after "time" β, to not exit $\mathfrak{D}_s$ and survive its encounters with the objects. The survival probability of any particular Brownian bridge in $\mathfrak{D}_s$ is given by $p_s(\ell_\beta(\mathbf{x}))=\exp[-\int_0^\beta V_s(\mathbf{x}_t)\,\mathrm{d}t]$, where V_s is the sum of local potentials representing the objects in $\mathfrak{D}_s$. Dirichlet boundary conditions are imposed by setting $p_s=0$ if the loop crosses the surface and $p_s=1$ if it does not.

The contribution to $\tilde{\phi}^{(N)}(\beta)$ of a loop $\ell_\beta^{(\tau)}(\mathbf{x})$ that remains within $\mathfrak{D}_\emptyset$ and encounters all objects of $\tau\subsetneq\{1\dots N\}$ and no others is

$$\sum_{\substack{s\in\mathfrak{P}_N\\ s\cap\tau=\gamma}}p_\gamma(-1)^{N-|s|}=\sum_{\gamma\in\mathfrak{P}(\tau)}p_\gamma\sum_{s=|\gamma|}^{N-|\tau|+|\gamma|}(-1)^{N-s}\binom{N-|\tau|}{s-|\gamma|}=0. \tag{14}$$

[1]A standard Brownian bridge $\ell_\beta(\mathbf{x})=\{\mathbf{x}+\sqrt{\beta}(\mathbf{W}(t)-t\mathbf{W}(1));0\leqslant t\leqslant 1\}$ is generated by a standard d-dimensional Wiener process with stationary and independent increments for which $\mathbf{W}(t>0)$ is normally distributed with variance td and vanishing average.

It does not contribute to $\tilde{\phi}^{(N)}(\beta)$ whatever the survival probabilities p_γ. Only loops that touch all N objects ($\tau = \{1,\ldots,N\}$) contribute to the alternating sum in eq. (5) and we have that

$$\tilde{\phi}^{(N)}(\beta) = (-1)^N \int_{\mathfrak{D}_\emptyset} \frac{\mathrm{d}\mathbf{x}}{(2\pi\beta)^{d/2}} \tilde{\mathcal{P}}^{(N)}[\mathbf{x};\beta], \tag{15}$$

where $\tilde{\mathcal{P}}^{(N)}[\mathbf{x};\beta]$ is the probability that a standard Brownian bridge starting at $\mathbf{x}$ and returning to $\mathbf{x}$ after "time" β does not exit $\mathfrak{D}_\emptyset$ and is killed by *every* one of the N objects. A Brownian bridge is killed by every one of N objects with probability,

$$p\text{ (killed by every one of N objects)} = \sum_{\gamma\in\mathfrak{P}_N} (-1)^{|\gamma|} p_\gamma. \tag{16}$$

Equation (16) is the extension to N objects of the statement,

$$p\text{ (killed by O}_1\text{ \textit{and} killed by O}_2) = (1-p_1)+(1-p_2)-(1-p_{12}) = p_\emptyset - p_1 - p_2 + p_{12}.$$

Note that eqs. (15) and (16) do not require survival probabilities to be independent ($p_{12}=p_1p_2$), which is true only for mutually disjoint objects described by potentials that do not overlap. Since $\tilde{\mathcal{P}}^{(N)}$ is a positive probability, the factor of $(-1)^N$ in eq. (15) determines the sign of $\tilde{\phi}^{(N)}(\beta)$. (For Dirichlet boundary conditions on all objects, the sign arises because paths that touch all N objects contribute only to $\phi_\emptyset(\beta)$.) The irreducible N-body Casimir energy of a scalar field interacting by local (positive) potentials with N objects that have no *common* intersection thus is finite and satisfies

$$(-1)^N \tilde{\mathcal{E}}^{(N)} < 0. \tag{17}$$

It is remarkable that the sign of $\tilde{\mathcal{E}}^{(N)}$ in this case depends only on the number of objects: for scattering by local potentials, the irreducible scalar two-body Casimir energy, in particular, is negative, independent of any symmetry [11]. Equation (17) also holds in the limit of Dirichlet boundary conditions. But eq. (17) evidently [24] is not correct for boundary conditions, like Neumann's, that cannot be described by potentials. Also, the sign of the N-body Casimir energy does not, of itself, determine whether Casimir *forces* are attractive or repulsive. The subtractions in equation (12) define an irreducible part of the vacuum energy and it is important to correctly interpret this difference of vacuum energies. The finite N-body Casimir energy considered here is the irreducible N-body *correction* to the vacuum energy that remains when *all* irreducible M-body vacuum energies with $0 \leqslant M < N$ have been accounted for. The latter may themselves be finite but very often are not, and the sign of $\tilde{\mathcal{E}}^{(N)}$ determined by eq. (17) is that of the irreducible N-body part only. The irreducible N-body Casimir energy depends strongly on what one considers to be the N objects and, in general, does not coincide with the work needed to assemble these objects from infinity.

For a scalar field, eq. (15) interprets $\tilde{\phi}^{(N)}(\beta)$ as a probability for a random walk that satisfies certain geometric conditions. Since they have to touch N objects that have no common intersection, Brownian bridges that contribute in eq. (15) necessarily are of *finite* length. The probability $\tilde{\mathcal{P}}^{(N)}[\mathbf{x};\beta]$ thus is bounded from above by the shortest closed classical path of length $\ell_{\min}$ that just touches *all* the objects,

$$0 \leqslant \tilde{\mathcal{P}}^{(N)}[\mathbf{x};\beta] \leqslant e^{-\ell_{\min}^2/(2\beta)}. \tag{18}$$

In agreement with the more general argument given previously, eq. (18) implies that the asymptotic power series in β of $\tilde{\phi}^{(N)}(\beta\sim 0)$ vanishes to all orders.

Examples. – Consider the example of a scalar field in $\mathcal{R}^d$ satisfying Dirichlet boundary conditions on $(d+1)$ intersecting, $(d-1)$-dimensional hyper-planes. In this case $\tilde{\mathcal{E}}^{(d+1)}$ indeed is the work required to adiabatically move the last hyper-plane into position from infinity: $\tilde{\mathcal{E}}^{(d+1)}$ vanishes as the volume enclosed by the hyper-planes becomes infinite and depends continuously on their position. These are simple consequences of the smoothness and continuity of the probability for Brownian bridges to cross all hyper-planes in time β. Equation (12) implies that (infinite) hyper-planes forming a simplex, such as a triangle ($d=2$) or a pyramid ($d=3$), tend to repel (triangle) in even and to attract (pyramid) in odd dimensional spaces. The contributions to Casimir energies from interior modes of domains with generalized reflection symmetries have been obtained analytically as well as numerically [25–27]. None of these references consider only a finite irreducible part of the three-body Casimir energy and some of the results disagree. The world-line method [7,15,21] outlined above could provide fairly accurate numerical estimates for the irreducible N-body part of the Casimir energy of a scalar field with Dirichlet boundary conditions on any generic set of N intersecting hyper-planes without restriction to the contributions from interior modes only. The irreducible three-body Casimir energies of some weakly interacting intersecting objects have been analytically found [28] to be positive and finite.

The Casimir energy for 2^d pairwise parallel $(d-1)$-dimensional hyper-planes forming a multi-dimensional tic-tac-toe–like pattern in $\mathcal{R}^d$ that encloses an inner hyper-rectangle with dimensions $\ell_1\times\cdots\times\ell_d$ was previously considered in [29], but without separating irreducible parts. The irreducible 2^d-body Casimir energy $\tilde{\mathcal{E}}_\#^{2^d}$ for a scalar field satisfying Dirichlet conditions on all the hyper-planes has the simple form

$$\tilde{\mathcal{E}}_\#^{2^d} = -\frac{\hbar c\Gamma[(d+1)/2]}{4\pi^{(d+1)/2}} \sum_{n_1=1}^\infty \cdots \sum_{n_d=1}^\infty \frac{\mathcal{V}_\#}{L^{d+1}(\mathbf{n})}, \tag{19}$$

where $\mathcal{V}_\# = \prod_{j=1}^d \ell_j$ is the volume of the enclosed hyper-rectangle and $L(\mathbf{n}) = \sqrt{\sum_{j=1}^d n_j^2 \ell_j^2}$ is half the length of a classical periodic orbit in its *interior* that reflects n_j times off the j-th pair of parallel hyper-planes. Only classical periodic orbits that touch all hyper-planes contribute to $\tilde{\mathcal{E}}_\#^{2^d}$. Consistent with the arguments above, $\tilde{\mathcal{E}}_\#^{2^d}$ is negative and finite and remains so in the limit where one or more dimensions of the rectangle vanish and up to $(d-1)$ pairs of hyper-planes coincide. Note that $\tilde{\mathcal{E}}_\#^{2^d}(\ell_k \to 0) = \frac{1}{2}\tilde{\mathcal{E}}_\#^{2^{d-1}}$. When *any* dimension of the rectangle becomes large, $\tilde{\mathcal{E}}_\#^{2^d}$ vanishes. As mentioned before, this analyticity in the position of the objects is expected in the world-line description. It is one of the more interesting characteristics of the irreducible N-body Casimir energies defined by eqs. (3) and (5). The irreducible Casimir energy of eq. (19) has an intrinsic meaning that does not involve the (divergent) subtracted contributions: $\tilde{\mathcal{E}}_\#^{2^d}$ determines higher-order derivatives of the vacuum energy in much the same manner as the original Casimir energy gives the force between two parallel plates,

$$\partial_{\ell_1}\partial_{\ell_2}\dots\partial_{\ell_d}\mathcal{E}_\# = \partial_{\ell_1}\partial_{\ell_2}\dots\partial_{\ell_d}\tilde{\mathcal{E}}_\#^{2^d}. \tag{20}$$

Equation (20) holds because the subtracted contributions do not depend on *all* the dimensions of the rectangle. For $d=2$ pairs of parallel plates eq. (20) implies that this irreducible four-body contribution fully describes certain stability derivatives of the vacuum energy. In general, irreducible N-body Casimir energies give certain, sometimes complicated, derivatives of the vacuum energy. Finite irreducible N-body Casimir energies do not depend sensitively on the (quantum) description of intersecting objects at high scattering energy and the corresponding derivatives of the vacuum energy are reliably estimated by low-energy effective models that describe the interaction with the objects in terms of potentials or boundary conditions. The price one pays for this simplified description is that material-dependent (ultraviolet-sensitive) integration constants remain undetermined.

An estimate of the magnitude of irreducible three-body electromagnetic Casimir forces is provided by the Casimir-Polder force on a polarizable atom due to a bilayer. Using the results of [30], fig. 2 compares the irreducible electromagnetic three-body Casimir force on an atom with the sum of irreducible two-body Casimir forces for a Si/SiO_2 bilayer. It is well known that Casimir forces are not additive and the three-body correction is not negligible in this example. It *reduces* the overall attractive force by almost 15% for atoms that are about 10 times the thickness of the SiO_2 layer from the surface.

Irreducible Casimir energies in the multiple scattering expansion. – To address more complicated geometries in the electromagnetic case, a representation of irreducible Casimir energies in terms of one-body scattering operators is required [12]. For three bodies it is obtained using the generating functional approach

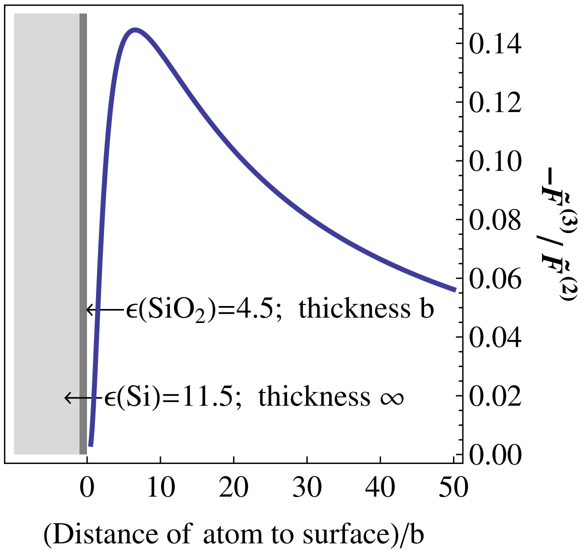

Fig. 2: (Colour on-line) Ratio of the irreducible three-body Casimir force to the sum of irreducible two-body Casimir forces on an atom near a Si/SiO_2 bilayer. The distance of the atom from the surface is measured in units of the thickness, b, of the SiO_2 layer.

of [31]. In the notation of [13,28], the irreducible three-body Casimir energy $\tilde{\mathcal{E}}^{(3)}$ in terms of the free-, 1-, 2- and 3-body Green's functions is

$$\begin{aligned}\tilde{\mathcal{E}}^{(3)} &= \frac{i}{2\tau}\mathrm{Tr}(\ln G_{123} - \ln G_{12} - \ln G_{23} - \ln G_{13} \\ &\quad + \ln G_1 + \ln G_2 + \ln G_3 - \ln G_\emptyset) \\ &= \frac{-i}{2\tau}\mathrm{Tr}(\ln \tilde{G}_1\tilde{G}_{123}^{-1}\tilde{G}_{23} - \ln \tilde{G}_1\tilde{G}_{12}^{-1}\tilde{G}_2 - \ln \tilde{G}_1\tilde{G}_{13}^{-1}\tilde{G}_3),\end{aligned} \tag{21}$$

where $G_s = G_\emptyset \tilde{G}_s$ is the Green's function for the domain $\mathfrak{D}_s$. The trace is over space and time, with τ here denoting the temporal extent. Using $\tilde{G}_{ij}^{-1} = \tilde{G}_i^{-1} + \tilde{G}_j^{-1} - \mathbb{1}$ and $\tilde{G}_{123}^{-1} = \tilde{G}_1^{-1} + \tilde{G}_{23}^{-1} - \mathbb{1}$ with $\tilde{G}_i = \mathbb{1} - \tilde{T}_i$, the irreducible three-body Casimir energy of eq. (21) expressed by one-body scattering matrices T_i finally is

$$\begin{aligned}\tilde{\mathcal{E}}^{(3)} &= \frac{-i}{2\tau}\mathrm{Tr}(\ln[\mathbb{1} - \tilde{T}_1(\mathbb{1} - \tilde{G}_{23})] + \ln[X_{12}] - \ln[X_{13}]) \\ &= \frac{-i}{2\tau}\mathrm{Tr}\ln[\mathbb{1} - X_{12}\tilde{T}_1(\tilde{T}_2\tilde{T}_1\tilde{T}_3 - \tilde{G}_2\tilde{T}_3X_{23}\tilde{T}_2 \\ &\qquad - \tilde{G}_3\tilde{T}_2X_{32}\tilde{T}_3)X_{13}].\end{aligned} \tag{22}$$

Here $\tilde{T}_i = T_i G_\emptyset = (\mathbb{1} - \tilde{G}_i)$, with $G_\emptyset$ the Green's function for the domain $\mathfrak{D}_\emptyset$ with no objects inserted. The operators X_{ij} satisfy the integral equation

$$X_{ij}(\mathbb{1} - \tilde{T}_i\tilde{T}_j) = \mathbb{1}. \tag{23}$$

The expression in eq. (22) differs from that given in [12] insofar as all two-body interactions have been subtracted. The irreducible three-body Casimir energy of eq. (22) is continuous in the position of the three objects and remains finite when two of them overlap and the corresponding two-body contribution to the vacuum energy diverges. In fact, every term in eq. (22) requires scattering off all three objects and is finite. Using the multiple scattering formalism we have calculated [28] the three-body correction of eq. (22) to the Casimir energy of three semi-transparent parallel plates. We verified that the irreducible three-body Casimir energy remains finite when any two of the three plates coincide. The description of irreducible N-body Casimir energies by scattering matrices defines them for any local field theory and, in particular, for the physically interesting electromagnetic case.

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REFERENCES

[1] Schaden M. and Spruch L., *Phys. Rev. A*, **58** (1998) 935; *Phys. Rev. Lett.*, **84** (2000) 459; *Phys. Rev. A*, **65** (2002) 022108.

[2] Mazzitelli F. D., Sanchez M. J., Scoccola N. N. and von Stecher J., *Phys. Rev. A*, **67** (2003) 013807.

[3] Bulgac A., Magierski P. and Wirzba A., *Phys. Rev. D*, **73** (2006) 025007.

[4] Kabat D., Karabali D. and Nair V. P., *Phys. Rev. D*, **81** (2010) 125013.

[5] Graham N., Jaffe R. L., Khemani V., Quandt M., Scandurra M. and Weigel H., *Nucl. Phys. B*, **645** (2002) 49; Graham N., Jaffe R. L., Quandt M., Schröder O. and Weigel H., *Nucl. Phys. B*, **677** (2004) 379; Schröder O., Scardicchio A. and Jaffe R. L., *Phys. Rev. A*, **72** (2005) 012105.

[6] Emig T., Hanke A., Golestanian R. and Kardar M., *Phys. Rev. Lett.*, **87** (2001) 260402; Büscher R. and Emig T., *Phys. Rev. Lett.*, **94** (2005) 133901; Emig T., Graham N., Jaffe R. L. and Kardar M., *Phys. Rev. Lett.*, **99** (2007) 170403; Reid M. T. H., Rodriguez A. W., White J. and Johnson S. G., *Phys. Rev. Lett.*, **103** (2009) 040401; Graham N., Shpunt A., Emig T., Rahi S. J., Jaffe R. L. and Kardar M., *Phys. Rev. D*, **81** (2010) 061701; Maghrebi M. F., Rahi S. J., Emig T., Graham N., Jaffe R. L. and Kardar M., preprint, arXiv:1010.3223v1, to be published in *Proc. Natl. Acad. Sci. U.S.A.* (2011) doi:10.1073/pnas.1018079108.

[7] Gies H. and Langfeld K., *Int. J. Mod. Phys. A*, **17** (2002) 966; Gies H., Langfeld K. and Moyaerts L., *JHEP*, **0306** (2003) 018; Gies H. and Klingmüller K., *Phys. Rev. Lett.*, **96** (2006) 220401; Weber A. and Gies H., *Phys. Rev. Lett.*, **105** (2010) 040403; Gies H. and Klingmüller K., *Phys. Rev. Lett.*, **97** (2006) 220405.

[8] Lambrecht A. and Marachevsky V. N., *Phys. Rev. Lett.*, **101** (2008) 160403; *Int. J. Mod. Phys. A*, **24** (2009) 1789; Chiu H.-C., Klimchitskaya G. L., Marachevsky V. N., Mostepanenko V. M. and Mohideen U., *Phys. Rev. B*, **81** (2010) 115417.

[9] Balian R. B. and Bloch C., *Ann. Phys. (N.Y.)*, **60** (1970) 401; **63** (1971) 592; **64** (1971) 271; *Errata*, **84** (1974) 559; **69** (1972) 76; **85** (1974) 514.

[10] Balian R. and Duplantier B., *Ann. Phys. (N.Y.)*, **104** (1977) 300; **112** (1978) 165; Balian R. and Duplantier B., in *Recent Developments in Gravitational Physics: Institute of Physics Conference Series*, edited by Ciufiolini A. *et al.* (Taylor & Francis, Boca Raton) 2004 Vol. **176**, p. 1, arXiv:quant-ph/0408124.

[11] Kenneth O. and Klich I., *Phys. Rev. Lett.*, **97** (2006) 160401; Bachas C. P., *J. Phys. A*, **40** (2007) 9089.

[12] Emig T., Graham N., Jaffe R. L. and Kardar M., *Phys. Rev. D*, **77** (2008) 025005, see eq. (III.27); Emig T. and Jaffe R. L., *J. Phys. A*, **41** (2008) 164001.

[13] Cavero-Pelaez I., Milton K. A., Parashar P. and Shajesh K. V., *Phys. Rev. D*, **78** (2008) 065018.

[14] Fulling S. A., Milton K. A., Parashar P., Romeo A., Shajesh K. V. and Wagner J., *Phys. Rev. D*, **76** (2007) 25004; Estrada R., Fulling S. A., Liu Z., Kaplan L., Kirsten K. and Milton K. A., *J. Phys. A*, **41** (2008) 164055.

[15] Schaden M., *Phys. Rev. Lett.*, **102** (2009) 060402.

[16] Feynman R. P. and Hibbs A. R., *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York) 1965; Kac M., *Am. Math. Mon.*, **73** (1966) 1.

[17] Greiner P., *Arch. Ration. Mech. Anal.*, **41** (1971) 163.

[18] Gilkey P. B., *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem* (Publish or Perish, Wilmington) 1984 and (CRC Press, Boca Raton) 1995.

[19] Kirsten K., *Spectral functions in Mathematics and Physics* (Chapman & Hall/CRC Press, Boca Raton) 2002.

[20] Vassilevich D. V., *Phys. Rep.*, **388** (2003) 279.

[21] Schaden M., *Phys. Rev. A*, **79** (2009) 052105.

[22] Schaden M., *Phys. Rev. A*, **73** (2006) 042102; **82** (2010) 022113.

[23] Fulling S. A., Kaplan L., Kirsten K., Liu Z. H. and Milton K. A., *J. Phys. A*, **42** (2009) 155402.

[24] Boyer T. H., *Phys. Rev. A*, **9** (1974) 2078; Hushwater V., *Am. J. Phys.*, **65** (1997) 381; Schaden M. and Spruch L., *Phys. Rev. A*, **58** (1998) 935.

[25] Dowker J. S., *Class. Quantum Grav.*, **23** (2006) 2771.

[26] Ahmedov H. and Duru I. H., *J. Math. Phys.*, **46** (2005) 022303; 022304; *Phys. At. Nuclei*, **68** (2005) 1621; Ahmedov H., *J. Phys. A*, **40** (2007) 10611.

[27] Abalo E. K., Milton K. A. and Kaplan L., *Phys. Rev. D*, **82** (2010) 125007.

[28] Shajesh K. V. and Schaden M., preprint, arXiv:1103.3048, submitted to *Phys. Rev. D*.

[29] Svaiter N. F. and Svaiter B. F., *J. Phys. A*, **25** (1992) 979.

[30] Zhou F and Spruch L., *Phys. Rev. A*, **52** (1995) 297; Salem R., Japha Y., Chab J., Hadad B., Keil M., Milton K. A. and Folman R., *New J. Phys.*, **12** (2010) 023039, appendix A.

[31] Bordag M., Robaschik D. and Wieczorek E., *Ann. Phys. (N.Y.)*, **165** (1985) 192; Robaschik D., Scharnhorst K. and Wieczorek E., *Ann. Phys. (N.Y.)*, **174** (1987) 401; Bordag M., Hennig D. and Robaschik D., *J. Phys. A*, **25** (1992) 4483; Bordag M., *Phys. Rev. D*, **73** (2006) 125018.