PAPER

An efficient algorithm for cooling particles in a controlled light standing wave

IOP Publishing Journal of Physics B: Atomic, Molecular and Optical Physics
J. Phys. B: At. Mol. Opt. Phys. **49** (2016) 085001 (8pp) doi:10.1088/0953-4075/49/8/085001

An efficient algorithm for cooling particles in a controlled light standing wave

D A Ivanov and T Yu Ivanova

St. Petersburg State University, Ulianovskaya 3, Petrodvorets, St. Petersburg, 198504, Russia

E-mail: ivanov-den@yandex.ru

Received 2 November 2015, revised 1 February 2016
Accepted for publication 29 February 2016
Published 5 April 2016

Abstract

A new approach to feedback cooling of many-particle ensembles is considered. The homodyne detection of counter-propagating probe beams is shown to provide enough information to realize an efficient 'bang-bang' cooling algorithm. The feasibility of the method is demonstrated via numerical Monte Carlo simulations of classical trajectories of the particles taking into account the spontaneous emission noise. The numerical analysis and approximate theoretical treatment predict the cooling rate as being inverse proportional to the square root of the number of particles in the ensemble.

Keywords: feedback cooling, collective effects, dipole potential

(Some figures may appear in colour only in the online journal)

1. Introduction

The application of laser cooling to particles with a complex structure of energy levels is limited due to the absence of cycle transitions [1]. The standard approach to overcome this difficulty is to use re-pumping lasers, as in recent successive molecular cooling experiments [2, 3]. Another approach is to use dipole force for cooling, thereby avoiding internal transitions. The mechanism of energy extraction in this case is not obvious. In one group of such methods the motion of particles is dipole-coupled to an optical cavity field [4–8] with subsequent energy loss via cavity leaking. The other group of methods is feedback cooling, where the loss channel is similar to cavity cooling, but mediated by an electronic feedback circuit. There are many feedback cooling proposals and experiments [9–14].

In a recent paper [15] we proposed to combine cavity cooling and feedback. In particular, we considered the photodetection of light back-scattered from a sample of particles to adjust the strength of the periodic optical potential. It has been shown that cooling takes place if the optical potential is proportional to the integral of the measured signal. The algorithm involved in the integration is a copy of the process that takes place in the cavity also without additional electronic feedback. The introduction of feedback, however, allows for more flexible processing of the measured signal and thus, possibly, a more efficient cooling algorithm. We will explore this possibility.

It will be shown that an efficient cooling algorithm, discussed in [12] for a single atom case, can be adopted to the feedback setup presented in [15]. In the algorithm, an optical potential with a fixed depth is applied to the particles only if they on average approach the potential maximums and is switched off otherwise. Such a control type is often referred to as a 'bang-bang' control.

Numerical simulations of particle motion subjected to feedback will be shown to demonstrate the rapid decrease of the ensemble energy. While performing the simulations, we take into account the spontaneous emission to vacuum modes that results in the random momentum kicks and heats the sample. This effect will be shown to decrease the rate of cooling.

Another important question that arises when dealing with the many-particle systems is the dependence of cooling on the number of particles. Unfortunately, since the cooling method we discuss here is based on the measurement of the collective observable, the cooling rate decreases with the number of particles. The rate is inversely proportional to the square root of the particle number. However, the estimations we made on the basis of the numerical simulations show that even for rather large ensembles the method demonstrates a reasonable cooling time.

2. Cooling algorithm and setup scheme

In the scheme that we are going to discuss the cooling will be achieved via appropriate control of the intensity I of the fields forming the optical standing wave potential $U = U_0 I \sin^2(kx)$, where k is the wave number of the fields and U_0 is the constant factor for the fixed detuning [16]. The motion of the particles will be restricted to the single dimension along the x axis. The classical equations of motion for the ith particle of mass m are then given by

$$\begin{aligned} \frac{\mathrm{d}p_i}{\mathrm{d}t} &= -U_0 I k \sin(2kx_i), \\ \frac{\mathrm{d}x_i}{\mathrm{d}t} &= \frac{p_i}{m}. \end{aligned} \tag{1}$$

To derive the control law we consider the change of the kinetic energy of the ensemble of N particles

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{kin}} &= \sum_{i=1}^{N}\frac{p_i}{m}\frac{\mathrm{d}p_i}{\mathrm{d}t} = -U_0 I k \sum_{i=1}^{N}\frac{p_i}{m}\sin(2kx_i) \\ &= -\frac{U_0 I}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}\cos(2kx_i)\right). \end{aligned} \tag{2}$$

To compensate for the kinetic energy one should keep the derivative of the energy in equation (2) negative. This can be done by the appropriate control of the potential depth determined by the intensity I. Since $I \geqslant 0$, to obtain the net energy subtraction one should apply the potential only if $\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \cos(2kx_i) > 0$, otherwise the optical potential should be switched off. In this case the energy change according to equation (2) will be always negative or at least zero. Furthermore, to provide the maximal possible energy subtraction one should apply the maximal available intensity $I_{\max}$. The 'bang-bang' control law then reads

$$I = I_{\max}\theta\left(\sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\cos(2kx_i)\right), \tag{3}$$

where $\theta(x)$ is the Heaviside step function with $\theta(0) = 0$. A similar concept applied to a single atom has been proposed in [12]. Thus, the feedback implies the measurement of the collective quantity $\sum_{i=1}^{N}\cos(2kx_i)$. We will discuss below a possible physical realization of this cooling algorithm in a ring cavity modeling the particles as two-level atoms.

The two-level approximation should not be considered as a key assumption relevant for the validity of the discussed approach. Moreover, as will be seen below, the natural regime of the method is off-resonant interaction between the light and the particles, where the only important parameter is the polarizability of the particles. In general, the approach we discuss here is applicable to a broad range of particles including heavy nanoparticles, as discussed in [17, 18]. However, for the first experimental demonstration of the discussed approach it might be advantageous to use precooled alkali atoms with strong atom-field interaction provided by relatively small detuning. In this regime the heating due to spontaneous emission can be relevant. The two-level approximation used here allows us to take this effect into account in a simple way.

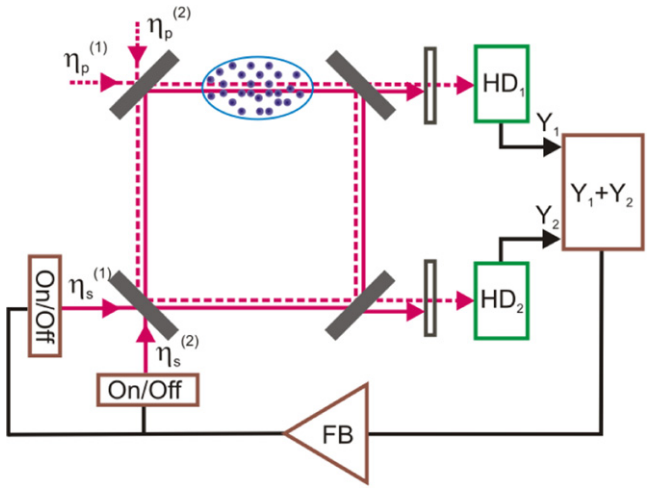

Figure 1. Setup scheme. The phase quadratures of the scattered probe fields $\eta_p^{(1)}$ and $\eta_p^{(2)}$ are measured with homodyne detectors (HDs) and combined to give the required collective observable. In the control circuit (FB) the time derivative of the measured signal is calculated and the signal to switch on/off the strong fields, $\eta_s^{(1)}$ and $\eta_s^{(2)}$, is generated.

As a model system we consider the atoms placed inside a ring cavity as shown in figure 1. The cavity is pumped with weak probe fields, $\eta_p^{(1)}$ and $\eta_p^{(2)}$, as well as with strong controlled fields, $\eta_s^{(1)}$ and $\eta_s^{(2)}$. The probe and controlled fields should be distinguishable (via polarization or small frequency detuning), but phase-locked. The atoms scatter photons between the counter-propagating probe (dashed lines) and strong (solid lines) modes. The scattered probe signals are measured using homodyne detectors HD_1 and HD_2. The results of these measurements are then combined to yield $\sum_{i=1}^{N}\cos(2kx_i)$ and processed in the circuit FB according to the algorithm described above to switch on/off the strong fields, $\eta_s^{(1)}$ and $\eta_s^{(2)}$, which provide the controlled optical potential for the atoms.

In a practical realization an additional, for example optical dipole, trap may be needed to keep the atoms inside the cavity. As follows from equation (3) for cold atoms, when the argument of the Θ-function is zero there is no potential ($I = 0$) applied to the atoms due to the feedback. Thus the feedback provides only cooling, not the trapping, mechanism. Simulating the cooling process, we will not include a trapping potential and will focus on the kinetic energy of the atoms leaving the spread of the cloud undefined. In a more experimentally-oriented consideration, an additional trapping potential can easily be included.

3. Mathematical model

The atomic system is modeled as a set of identical two-level atoms. The ground and excited states of the atoms are labeled as $|g\rangle$ and $|e\rangle$. The transition frequency between these

levels is ω_0. The internal dynamics of the ith atom is described by the operators $\sigma_i^+ = |e\rangle\langle g|$, $\sigma_i^- = |g\rangle\langle e|$ and $\sigma_i^z = |e\rangle\langle e| - |g\rangle\langle g|$ obeying commutation relations $[\sigma_i^-, \sigma_i^+] = -\sigma_i^z$ and $[\sigma_i^-, \sigma_i^z] = 2\sigma_i^-$.

The atoms interact with the counter-propagating modes of the ring cavity. There is no principal difference between the interaction of the atoms with the probe and the strong modes. Thus, we first derive the general equations for the atoms and two counter-propagating cavity modes and then apply specific approximations to the weak probe and to the strong modes. The two counter-propagating modes with the frequency ω_L are represented by the photon creation and annihilation operators $a_i^\dagger$ and a_i, where $i = 1, 2$ correspond to the clockwise and the counterclockwise modes, respectively. These cavity modes are excited by two external coherent fields with amplitudes $\eta^{(1)}$ and $\eta^{(2)}$.

The total Hamiltonian of the system can be written in the form

$$H = H_0 + H_p + H_{int} + H_R. \tag{4}$$

The Hamiltonian H_0 describes the free atoms and the free radiation fields and reads

$$H_0 = \frac{\hbar\omega_0}{2}\sum_i(\sigma_i^z + 1) + \sum_i \frac{p_i^2}{2m} + \sum_{l=1}^{2}\hbar\omega_L a_l^\dagger a_l. \tag{5}$$

H_p is responsible for the pumping of the cavity modes

$$H_p = -i\hbar\sum_{l=1}^{2}(\eta_l^* a_l - \eta_l a_l^*). \tag{6}$$

The H_{int} is the atom–light interaction Hamiltonian

$$H_{int} = -\hbar g\sum_i (a_1 e^{ik_L x_i} + a_2 e^{-ik_L x_i})\sigma_i^+ + \text{h.c.}\,, \tag{7}$$

where $g = id\sqrt{\omega_L/2\hbar\epsilon_0 V}$ describes the coupling of the atoms to the cavity fields, with d being the electric dipole moment of the atoms and V being the volume of the cavity.

In order to take into account the spontaneous emission and the corresponding heating effect, the atoms are additionally coupled to a reservoir of vacuum modes of frequencies ω_k with creation and annihilation operators $b_{\vec{k}s}^\dagger$ and $b_{\vec{k}s}$. The decay of the cavity modes 1 and 2 is due to their interaction with the vacuum field outside the cavity, described by bosonic modes $c1_{\vec{q}s}^\dagger$, $c1_{\vec{q}s}$ with frequencies $\omega_{\vec{q}_1}$ and $c2_{\vec{q}s}^\dagger$, $c2_{\vec{q}s}$ with frequencies $\omega_{\vec{q}_2}$. Thus the whole reservoir contribution to the Hamiltonian reads

$$\begin{aligned} H_R &= \sum_{\vec{k},s}\hbar\omega_k b_{\vec{k},s}^\dagger b_{\vec{k},s} - i\sum_j\sum_{\vec{k},s}(\hbar g_{\vec{k},s} b_{\vec{k},s} e^{ik_x x_j}\sigma_j^+ + \text{h.c.}) \\ &+ \sum_{\vec{q}_1,s}\hbar\omega_{q_1} c1_{\vec{q}_1,s}^\dagger c1_{\vec{q}_1,s} + \sum_{\vec{q}_2,s}\hbar\omega_{q_2} c2_{\vec{q}_2,s}^\dagger c2_{\vec{q}_2,s} \\ &- i\sum_{\vec{q}_1,s}(\hbar g_{\vec{q}_1,s} c1_{\vec{q}_1,s} a_1^\dagger + \text{h.c.}) \\ &- i\sum_{\vec{q}_2,s}(\hbar g_{\vec{q}_2,s} c2_{\vec{q}_2,s} a_2^\dagger + \text{h.c.}) \end{aligned} \tag{8}$$

The atomic coupling to the reservoir is described by the constant $g_{\vec{k}s} = i\sqrt{\omega_k/2\hbar\epsilon_0\mathcal{V}}(\vec{d}\cdot\vec{e}_{\vec{k}s})$, where $\vec{e}_{\vec{k},s}$ is the vacuum mode polarization vector and $\mathcal{V}$ is the external quantization volume. The interaction constants of the cavity modes with their reservoirs $g_{\vec{q}_{1,2},s}$ are assumed to be weakly dependent on the reservoir mode number.

Using the Hamiltonians (4)–(8) the Heisenberg–Langevin equations for the atomic internal and external degrees of freedom as well as for the cavity fields can be obtained [19] and read:

$$\begin{aligned} \dot\sigma_i^- &= -\left(i\omega_0 + \frac{\Gamma}{2}\right)\sigma_i^- - ig(a_1 e^{ik_L x_i} + a_2 e^{-ik_L x_i})\sigma_i^z + F_\sigma^{(i)}, \\ \dot\sigma_i^z &= -\Gamma(\sigma_i^z + 1) + 2ig(a_1 e^{ik_L x_i} + a_2 e^{-ik_L x_i})\sigma_i^+ \\ &\quad - 2ig^*(a_1^\dagger e^{-ik_L x_i} + a_2^\dagger e^{ik_L x_i})\sigma_i^- + F_{\sigma^z}^{(i)}, \\ \dot p_i &= i g\hbar k_L(a_1 e^{ik_L x_i} - a_2 e^{-ik_L x_i})\sigma_i^+ \\ &\quad - ig^*\hbar k_L(a_1^\dagger e^{-ik_L x_i} - a_2^\dagger e^{ik_L x})\sigma_i^- + F_p^{(i)}, \\ \dot x_i &= p_i/m, \\ \dot a_1 &= -i\omega_L a_1 - \kappa a_1 - ig^* e^{-ik_L x_i}\sigma_i^- + \eta_1 + F_a^{(1)}, \\ \dot a_2 &= -i\omega_L a_2 - \kappa a_2 - g^* e^{ik_L x_i}\sigma_i^- + \eta_2 + F_a^{(2)}, \end{aligned} \tag{9}$$

where the cavity amplitude decay rate $\kappa = \sum\pi g_{\vec{q},s}/\hbar^2\delta(\omega_L - \omega_{\vec{q}})$. $F_{\dots}^{(i)}(t)$ are the operator Langevin forces having the following correlation functions

$$\begin{aligned} \langle F_{\dots}^{(i)}(t)\rangle &= 0, \\ \langle F_\sigma^{(i)}(t)F_{\sigma^+}^{(i)}(t')\rangle &= \frac{\Gamma}{2}\delta(t-t'), \\ \langle F_{\sigma^z}^{(i)}(t)F_{\sigma^z}^{(i)}(t')\rangle &= \Gamma(\langle\sigma_i^z\rangle + 1)\delta(t-t'), \\ \langle F_\sigma^{(i)}(t)F_{\sigma^z}^{(i)}(t')\rangle &= \Gamma\langle\sigma_i^-\rangle\delta(t-t') \\ \langle F_p^{(i)}(t)F_p^{(i)}(t')\rangle &= \frac{1}{20\pi}\Gamma\hbar^2 k_0^2(\langle\sigma_i^z\rangle + 1)\delta(t-t'), \end{aligned} \tag{10}$$

where the spontaneous decay rate $\Gamma = d^2\omega_0^3/(3\pi\hbar\epsilon_0 c^3)$ has been introduced.

Following the standard approach [20] the operator Heisenberg–Langevin equations (9) are substituted with the c-number equations using the operator ordering: $a^\dagger$, σ^+, $\{x, p\}$, σ^z, σ^- and a, which is symmetric with respect to the atomic position and momentum. The operators are substituted with the corresponding c-numbers α^*, s^*, x, p, s^z, s and α.

Then we adiabatically eliminated atomic internal degrees of freedom assuming that their relaxation constant Γ is much larger than the relaxation rates of all other dynamical variables. We also neglect the quantum noise in these variables. Then the stationary solutions describing internal degrees of freedom read as

$$\begin{aligned} s_i &= \frac{-ig(\alpha_1 e^{ik_L x_i} + \alpha_2 e^{-ik_L x_i})}{i\delta + \Gamma/2}s_i^z, \\ s_i^z &= -\left[1 + \frac{2|g|^2(\alpha_1 + \alpha_2 e^{-2ik_L x_i})(\alpha_1^* + \alpha_2^* e^{2ik_L x_i})}{\delta^2 + (\Gamma/2)^2}\right]^{-1}, \end{aligned} \tag{11}$$

where $\delta = \omega_0 - \omega_L$ is the detuning between atomic transition and laser fields. Using this result and assuming that $\delta^2 + \Gamma^2/4 > 2|g|^2$ the evolution of the atomic momentum

is given by

$$\dot{p}_i = \frac{2i\hbar k_L |g|^2 \delta}{\delta^2 + \Gamma^2/4}(\alpha_2^* \alpha_1 e^{2ik_L x_i} - \alpha_1^* \alpha_2 e^{-2ik_L x_i}) + \frac{\hbar k_L |g|^2 \Gamma}{\delta^2 + \Gamma^2/4}(\alpha_1 \alpha_1^* - \alpha_2 \alpha_2^*) + \mathcal{F}_p^{(i)}. \quad (12)$$

In equation (12) the first term corresponds to the dipole force as it scales as $1/\delta$ for large detuning and the second term is responsible for the radiation pressure caused by possible imbalance of the clock- and counterclockwise modes amplitudes.

The general result for the mode amplitudes is given by the equation

$$\alpha_{1,2} = \frac{(i\delta + \Gamma/2)[|g|^2 N + \kappa(i\delta + \Gamma/2)]}{[|g|^2 N + \kappa(i\delta + \Gamma/2)]^2 - |g|^4 \sum_{l,m} e^{-2ik_L(x_l - x_m)}} \left[\eta_{1,2} - \frac{\eta_{2,1}|g|^2 \sum_j e^{\mp 2ik_L x_j}}{|g|^2 N + \kappa(i\delta + \Gamma/2)}\right]. \quad (13)$$

Below we consider rather weak coupling between the atoms and the cavity modes so that $\kappa|(i\delta + \Gamma/2)| \gg |g|^2 N$. This regime is addressed as it does not require micro-cavities and can be easily accessed in an experiment. Then equation (13) can be simplified to

$$\alpha_{1,2} = \sqrt{n_{1,2}} - \frac{\sqrt{n_{2,1}}\,|g|^2 \sum_i e^{\mp 2\, ik_L x_i}}{\kappa(i\delta + \Gamma/2)}, \quad (14)$$

where $n_{1,2} = (\eta_{1,2}/\kappa)^2$ denotes the number of photons in the probe or strong field modes in the absence of atoms.

In order to discuss the measurement of the required collective observable we make a step back and return to the quantum fields a_1 and a_2, which were replaced by classical amplitudes $\alpha_{1,2}$. The quantum analog of equation (14) has the same form, but instead of classical amplitude $\eta_{1,2}$ one should write operators $b_{1,2} = \eta_{1,2} + F_a^{(1,2)}$. The quantum Langevin forces are due to the vacuum noise entering the cavity and can be given in terms of the input fields as $F_a^{(1,2)} = \sqrt{2\kappa}\, a_{1,2}^{(IN)}$.

As discussed above, in order to perform the control one needs to know the time evolution of $\sum\cos(2k_L x_i)$. This quantity can be measured as follows. First, the Y-quadratures, $Y_{1,2}^{(OUT)} = (a_{1,2}^{(OUT)} - (a_{1,2}^{(OUT)})^\dagger)/2i$, of both of the output probe fields are measured via the HDs. This is schematically shown in figure 1. Assuming that the average driving field in both probe modes is the same, $\eta_1 = \eta_2 = \eta$, these quadratures read

$$Y_{1,2}^{(OUT)} = Y_{1,2}^{(IN)} - \frac{|g|^2}{2\kappa^2(\delta^2 + \Gamma^2/4)}[2\kappa(\Xi\Gamma/2 - \Theta\delta)Y_{2,1}^{(IN)} - (\sqrt{2\kappa}\,\eta + 2\kappa X_{2,1}^{(IN)})(\Theta\Gamma/2 + \Xi\delta)], \quad (15)$$

where we used the shortened notations $\Xi = \sum\cos(2k_L x_i)$ and $\Theta = \sum\sin(2k_L x_i)$. The input field X-quadratures $X_{1,2}^{(IN)} = (a_{1,2}^{(IN)} + (a_{1,2}^{(IN)})^\dagger)/2$ have also been introduced. The well-known input-output relation [21] has also been used to link the input, output and intracavity fields. It is easy to see that the sought many-atom variable Ξ can be obtained taking the sum of the two Y-quadratures: $Q \equiv Y_1^{(OUT)} + Y_2^{(OUT)}$. Indeed the quantum average of this sum reads

$$\langle Q \rangle = \frac{2\,|g|^2\sqrt{2\kappa}\,\eta\delta}{\kappa^2(\delta^2 + \Gamma^2/4)} \sum_i \cos(2k_L x_i). \quad (16)$$

The use of the ring cavity and homodyne detection may look like an unnecessary complication since, as discussed in [10, 12], the linear cavity transmission can be measured to determine Ξ. This is true if the number of atoms is fixed. Generalizing the approach of [12] to the many-atom case we obtain that the transmission signal depends on $\sum_i \cos^2(k_L x_i)$. The derivative of this quantity is proportional to the required derivative of $\sum_i \cos(2k_L x_i)$ only if the number of atoms is fixed. Otherwise, the atom-number fluctuations deteriorate the measured signal. This effect does not appear in the case of homodyne detection, where the required quantity is measured directly.

The measurement noise poses a limit to the performance of the feedback cooling. The dominant source of the measurement noise is the photon number fluctuations in the probe fields. The noise spectrum due to the quantum fluctuations of the measured quantity can be found using equation (15). We separate the noise due to the quantum light fluctuations and the variations of the measured quantity due to the atomic motion. First, we assume that the atoms do not move during the measurement time and calculate the correlation function taking the average over the quantum state of the probe light. The corresponding noise spectrum is calculated by performing the Fourier transform of this correlation function. For the detected quantity Q we obtain the following noise spectrum

$$S_q(\omega) = \frac{1}{4}\left[1 - \frac{|g|^2(\Gamma\Xi - 2\delta\Theta)}{\kappa(\Gamma^2/4 + \delta^2)}\right]^2 + \frac{1}{4}\left[1 - \frac{|g|^2(\Gamma\Xi + 2\delta\Theta)}{\kappa(\Gamma^2/4 + \delta^2)}\right]^2 + \frac{1}{4}\frac{|g|^4(\Gamma\Theta - 2\delta\Xi)^2}{\kappa^2(\Gamma^2/4 + \delta^2)^2} + \frac{1}{4}\frac{|g|^4(\Gamma\Theta + 2\delta\Xi)^2}{\kappa^2(\Gamma^2/4 + \delta^2)^2}. \quad (17)$$

Here we assumed that the input fields are in the vacuum state so that all the quadrature correlations equal $1/4\delta(t - t')$. In the regime of weak atom–field coupling as described above we assumed $\kappa|\Gamma/2 + i\delta| \gg |g|^2 N$. Taking this inequality into account in equation (17) we see that the expression for the quantum noise spectrum simplifies to the vacuum noise limit

$$S_q(\omega) \approx \frac{1}{2}. \quad (18)$$

Let us now compare the quantum measurement noise in equation (18) with the typical variations of the signal due to the motion of the atoms. Using the average signal, equation (16), we calculate the spectrum of the fluctuations of this signal due to the chaotic motion of the atoms. For the sake of simplicity we assume that the atoms are independent and characterized by identical single-atom distribution functions. These single-atom distribution functions will be

approximated as

$$f(x,p)=\frac{1}{\pi\sigma\sqrt{2mk_{\mathrm{B}}T}}\exp\left[-\frac{p^2}{2mk_{\mathrm{B}}T}-\frac{x^2}{\sigma^2}\right], \qquad (19)$$

where σ is the spatial localization. The other parameters in equation (19) are the atomic mass m, the ensemble temperature T, and the Boltzmann constant k_{B}. The calculation of the correlation function $\overline{\delta Q(t)\delta Q(t')}$ and the subsequent Fourier transform give the following spectrum of the fluctuations due to the atomic motion

$$S_{\mathrm{cl}}(\omega)=\frac{8\,|g|^4\delta^2\eta^2N}{\kappa^3(\Gamma^2/4+\delta^2)^2}\frac{(1-\mathrm{e}^{-4k_{\mathrm{L}}^2\sigma^2})^2}{k_{\mathrm{L}}\overline{V}}\mathrm{e}^{-\frac{\omega^2}{\pi k_{\mathrm{L}}^2\overline{V}^2}}. \qquad (20)$$

In this expression we used the average velocity of the atoms $\overline{V}=\sqrt{8k_{\mathrm{B}}T/(\pi m)}$ in the ensemble. The feedback will operate correctly if the 'classical' spectrum, equation (20), will be larger than the quantum white noise, equation (18), in some spectral interval. For simplicity we require that the classical fluctuations exceed the quantum noise for the zero frequency. Furthermore, we distinguish between uniformly distributed atomic ensemble with $\sigma\to\infty$ and the localized ensemble where the opposite limit holds, $\sigma\to 0$. The condition for the measurement to provide correct information on the atomic state for the uniformly distributed atoms then reads

$$\frac{8C^2\delta^2\kappa n_p}{N(\Gamma^2/4+\delta^2)}\frac{1}{k_{\mathrm{L}}\overline{V}}>\frac{1}{2}. \qquad (21)$$

Here the photon number in the probe field $n_p=(\eta/\kappa)^2$ and the atom-field coupling parameter $C\equiv|g|^2N/(\kappa\sqrt{\Gamma^2/4+\delta^2})\ll 1$ have been defined. This inequality allows to estimate the number of probe photons needed to correctly measure the atoms. It is seen that the critical parameter is $k_{\mathrm{L}}\overline{V}$, which is the inverse of the time an atom moves over a single period of the feedback potential. For high temperature and high atomic velocities the number of photons should be sufficiently large. If for example, $C\sim 0.1$ and $k_{\mathrm{L}}\overline{V}\sim\kappa$, then the required number of photons in the probe field will be $n_p>10N$.

This requirement changes if the atoms are well localized. For such ensembles the measurement condition reads

$$\frac{8C^2\delta^2\kappa n_p}{N(\Gamma^2/4+\delta^2)}\frac{k_{\mathrm{L}}^3\sigma^4}{\overline{V}}>\frac{1}{2}. \qquad (22)$$

This result shows that for tight localization, $\sigma\to 0$ the quantum noise can dominate if the atomic velocity is not sufficiently small. This can pose a limit to the operation of the feedback cooling. However, even in an unfavorable case of too small ratio $\sigma^4/\overline{V}$, the feasibility of the feedback cooling can be restored by choosing a sufficiently large number of the probe-field photons. Thus the quantum measurement noise should not spoil the feedback cooling at least in principle. Therefore in this paper we assume that the feedback cooling operates in the regime where the measurement noise can be neglected.

The role of the actuator is played by the pair of additional strong modes, which form the already-mentioned feedback potential. These fields also get scattered by the atoms, but the change of their amplitudes due to such scattering is very small compared to the unperturbed values, so in the following simulations we neglect this effect. Another effect that will also be ignored is the influence of the probe fields on the atomic momenta. This is done since the probe fields are assumed to be small compared to the strong controlled field.

The later assumption is not always valid as the cooling bang-bang algorithm prescribes a zero control field for the negative value of the derivative of $\sum\cos(2k_{\mathrm{L}}x_i)$. During the time periods when this condition is fulfilled the probe field is the only perturbation to the atoms. Thus, the intensity of the probe fields will contribute to the cooling limit. In this paper we concentrate on the efficiency of the feedback cooling at higher temperatures and neglect the effects that are relevant for the regime of low atomic energies. Being limited to quasi-classical motion of the atoms, we ignore the direct influence of the probe fields on the atomic momenta.

Thus, in the equations for the atomic momenta the optical fields are the strong controlled fields. In these equations we keep the Langevin forces, which allows us to take into account the heating due to spontaneous emission. The equation for the ith atom now reads

$$\dot{p}_i=-U_0n_{\mathrm{s}}\sin 2k_{\mathrm{L}}x_i+\mathcal{F}_{\mathrm{p}}^{(i)},$$
$$U_0=\frac{4\hbar k_{\mathrm{L}}\,|g|^2\delta}{\delta^2+\Gamma^2/4}, \qquad (23)$$

where the Langevin forces $\mathcal{F}_{\mathrm{p}}^{(i)}$ have the following correlations

$$\langle\mathcal{F}_{\mathrm{p}}^{(i)}(t)\mathcal{F}_{\mathrm{p}}^{(i)}(t')\rangle=D_0\delta(t-t'),$$
$$D_0=\frac{2\hbar^2k_0^2\Gamma\,|g|^2n_{\mathrm{s}}}{5\pi(\delta^2+\Gamma^2/4)}\cos^2(k_{\mathrm{L}}x_i). \qquad (24)$$

This result is obtained taking into account the 3D angular distribution of the spontaneously emitted photons and the fact that only the x-projection of the recoils contribute to the considered momentum.

4. Simulation results and discussions

We solve equations (23) numerically via direct Monte Carlo simulations of stochastic atomic trajectories. For time stepping we use the Euler–Maruyama method [22]. During these simulations we neglect possible measurement and processing errors that appear during the real-time evaluation of the derivative of $\sum\cos(2k_{\mathrm{L}}x_i)$. The intensities of the strong fields or their photon numbers $n_s=(\eta_s^{(1)}/\kappa)^2=(\eta_s^{(2)}/\kappa)^2$ are controlled as described by the algorithm in equation (3).

We assume that initially the atoms are homogeneously distributed over a single period of the optical potential produced by the strong control fields interference, that is over the length $\lambda_{\mathrm{L}}/2$. Due to the periodicity of the optical potential and the absence of the interaction between the atoms this assumption does not lead to the loss of generality. The initial momentum distribution is assumed to be Maxwellian with the

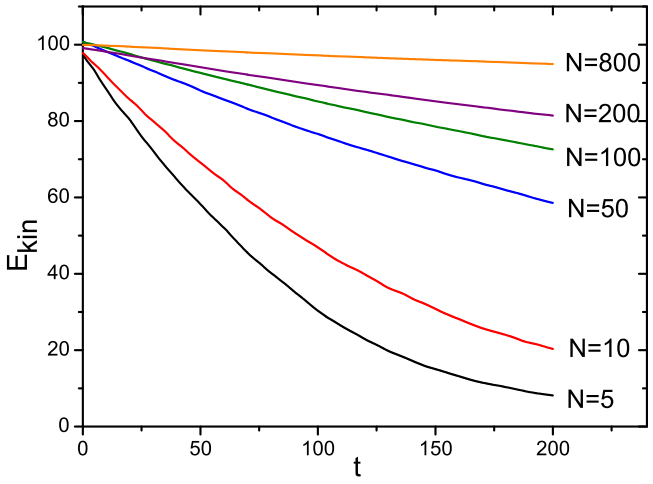

Figure 2. The evolution of the energy per particle in units of recoil energy for the ensembles with different numbers of atoms. The photon number in the controlled fields is 1000, the detuning is $\delta = 100\Gamma$. Time is measured in units of the excited state lifetime $1/\Gamma$.

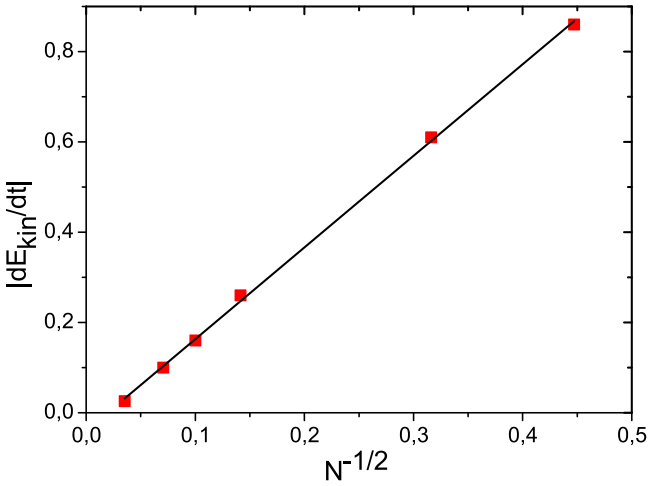

Figure 3. Dependence of the cooling rate for small t on the inverse square root of number of atoms.

temperature of the ensemble equal to 100 $\times$ recoil temperature $T_{\mathrm{rec}} = \hbar^2 k_{\mathrm{L}}^2/(2mk_{\mathrm{B}})$.

The simulation results for the fixed detuning $\delta = 100\Gamma$ and the fixed number of photons in the controlled fields $n_s = 1000$ are shown in figure 2. Different curves in this plot correspond to different numbers of atoms. Each curve is obtained by averaging over 500 realizations of initial conditions. The figure shows that small ensembles with fewer than 10 particles lose more than 80% of their energy in about 200 excited state lifetimes. The rate of cooling drops as the number of atoms increases. This is the consequence of the collective character of the cooling process, where there is no access to individual particles.

The curves corresponding to $N = 5$ and $N = 10$ in figure 2 show that the cooling slows down as the energy approaches zero. Thus, in order to characterize the cooling efficiency as a function of the number of atoms we compare the cooling rates at the beginning of the process, where (see figure 2) the energy decay is almost linear for all tested atomic numbers.

In figure 3 the slope of the energy time dependence is shown as a function of the inverse square root of the number of the atoms. The results of the simulations show that this dependence is well represented by the linear function, which means that the cooling rate is inverse proportional to the square root of the number of atoms. Extrapolating the results shown in figure 3 to an even larger number of atoms, one can estimate the cooling time of a rather big ensemble. For example, the ensemble of $N = 10^6$ atoms can be cooled, according to figure 3, in about 5×10^4 excited state lifetime cycles, which is about 1 millisecond. This estimation indicates that the cooling process can be of practical value. The obtained scaling of the cooling rate is typical for the methods based on the collective coupling of atoms to a cavity mode. See, for example [23], where the similar trend has been observed and discussed.

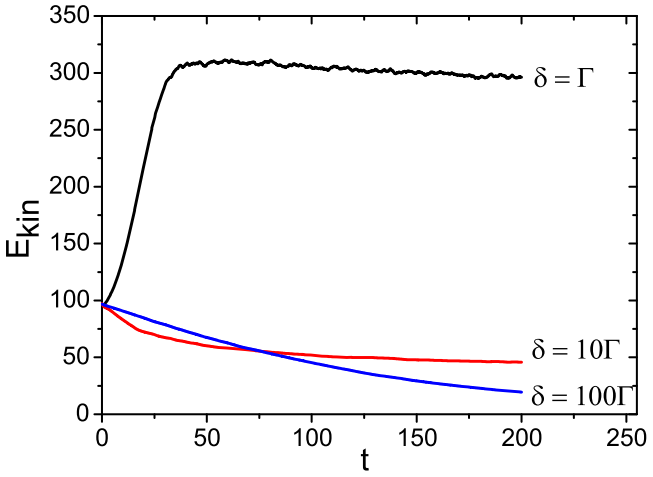

Figure 4. Evolution of the kinetic energy per atom for different values of the atom-field detuning δ.

For the purpose of practical implementations of the method it is interesting to consider the possibility of using rather small detuning δ. This allows for stronger coupling even for relatively small photon numbers. The noise due to spontaneous emission in this case will spoil the cooling as shown in figure 4.

The positive effect due to small δ is the faster drop of the kinetic energy in the beginning of the cooling process, compare the curves for $\delta = 100\Gamma$ and $\delta = 10\Gamma$ in figure 4. However for longer time the spontaneous emission limits the cooling rate. For extremely small detuning, as it is demonstrated by the case $\delta = \Gamma$ in figure 4, the cooling at the beginning of the process is changed to heating. Although a slow decrease of the energy takes place at later times, the regimes such as with $\delta = \Gamma$ are not practical and reasonably large values of the detuning should be recommended.

In order to illustrate physical reasons for some of the effects observed in numerical simulations we consider the limit when the atoms are already quite localized. In this case a rather simple and closed set of evolution equations for the

collective quantities can be derived via the harmonic approximation, i.e. taking $\sin(2k_{\mathrm{L}}x_i) \approx 2k_{\mathrm{L}}x_i$ and $\cos(k_{\mathrm{L}}x_i) \approx 1$. Let us consider the evolution of the following second moments:

$$K = \frac{1}{N}\sum_i \frac{p_i^2}{2m},$$
$$L = \frac{1}{N}\sum_i x_i^2,$$
$$G = \frac{2}{N}\sum_i x_i p_i. \tag{25}$$

Here K is the average kinetic energy per particle, L is the ensemble spatial localization, and G is the position-momentum correlation. Using the evolution equations (23) and taking into account the Ito rule when dealing with the stochastic terms in the momentum equations, we obtain the following evolution equations for the defined above averages

$$\begin{aligned} \frac{\mathrm{d}K(t)}{\mathrm{d}t} &= -[4k_{\mathrm{L}}U_0G(t) - D_0]n_s, \\ \frac{\mathrm{d}L(t)}{\mathrm{d}t} &= G(t), \\ \frac{\mathrm{d}G(t)}{\mathrm{d}t} &= 2K(t) - 8k_{\mathrm{L}}U_0L(t)n_s. \end{aligned} \tag{26}$$

The feedback algorithm that is considered here under harmonic approximation implies the measurement of the localization $L(t)$ and the application of the control action depending on the sign of its derivative: $L'(t) = G(t)$. In the proposed bang-bang type control we assume $n_s = 0$ if $G(t) \leqslant 0$ and $n_s = n_{\max}$ otherwise. Under these conditions there is a steady-state solution of equation (26) with $\bar{K} = 0$, $\bar{G} = 0$, $\bar{n}_s = 0$. The steady-state value of the localization $\bar{L}$ is not specified.

This simple result indicates that in the semi-classical regime with perfect measurement and signal processing capabilities the method can cool an ensemble down to zero temperature. This optimistic conclusion does not hold if the intensity of the probe field is taken into account. The probe will determine the minimal value of the field acting on the atoms if $G(t) \leqslant 0$. As it is seen from equation (26) there is no steady state solution in this case. The system will oscillate and will not reach the sought zero energy, $K = 0$. For large detuning, where the spontaneous emission can be completely ignored, the probe field n_p will determine the limiting kinetic energy $\bar{K} = 4k_{\mathrm{L}}U_0\bar{L}n_p$.

Note that contrary to feedback methods based on the idea of stochastic cooling [9, 11, 13] the efficient cooling here does not require remixing. This is seen from the fact that the zero energy steady state can be dynamically obtained without explicit or implicit use of direct atom–atom scattering. In this respect, the described approach, as well as the single-atom method of [10], is closer to the cavity cooling and self-organization [6] with the feedback due to the cavity enhanced by the carefully designed external electronic feedback.

Let us now apply equation (26) to estimate and qualitatively explain the dependence of the cooling rate on the number of atoms as shown in figure 3. The slope of the kinetic energy at the beginning of the evolution is given by the value of the derivative of the kinetic energy at $t = 0$, that is $[4k_{\mathrm{L}}U_0G(0) - D_0]n_s$. The dependence on the number of atoms is in the initial correlation $G(0)$. Taking into account the independence of the atoms in the initial thermal ensemble the variance of $G(0)$ reads

$$\langle\Delta[G(0)]^2\rangle = \frac{1}{N}\langle\Delta x^2\rangle_{\mathrm{th}}\langle\Delta p^2\rangle_{\mathrm{th}}. \tag{27}$$

Thus the magnitude of the initial value of the moment G can be estimated as $\sim 1/\sqrt{N}$, which also gives an approximate dependence of the cooling rate on the number of particles. Such a dependence reflects the collective character of the cooling process and poses certain limits on the size of the sample of particles that can be effectively cooled. The presented simple reasoning is confirmed by the simulation results, see figure 3.

Note that the steady-state zero energy can be obtained even in the presence of the noise due to spontaneous emission. The spontaneous emission can, however, greatly reduce the rate of cooling and even cause initial heating of the ensemble, if $4k_{\mathrm{L}}U_0G(0) - D_0 \leqslant 0$. This condition can be fulfilled if the number of atoms is large or the atom–field detuning δ is small. The simulation results, figure 4, showing the effect of the detuning are thus in qualitatively agreement with the conclusions of the harmonic approximation. Thus, simple analysis based on equation (26) can explain important features of the atomic evolution.

5. Conclusions

The efficient algorithm of feedback cooling of non-interacting many-particle ensembles has been discussed. The method is based on the measurement of the collective scattering of light from the ensemble. Thus, in this aspect this method is similar to systems considered in cavity cooling and self-organization. Contrary to these systems, the scattered signal in the discussed approach is processed electronically. This considerably increases the freedom in generating the control action based on the measurement result. In particular, we consider the feedback action proportional to the time derivative of the detected signal.

It is shown that the collective observable of the sample of which knowledge is required to obtain the feedback cooling is the sum of the cosines of atomic coordinates. This quantity can be measured via homodyne detection of the phase-quadrature of counter-propagating weak probe fields interacting with the cavity.

The numerical simulations of the evolution of the particles, performed with the spontaneous emission taken into account, demonstrate the obvious cooling effect. Although the cooling rate decreases for larger number of particles as the square root of this number, this effect does not seem to render the method impractical.

Simple analysis based on the harmonic approximation of the optical potential explains certain features of the particle

evolution qualitatively. The cooling of ensembles with a larger number of particles happens at a lower rate as the measured feedback signal, being the collective quantity, decreases as the square root of the number of particles.

Spontaneous emission was shown to decrease the cooling rate and in certain cases even cause initial heating of the ensemble. However, this process cannot affect the cooling limit as the light potential directly and consequently the spontaneous emission is switched off for the cold sample.

Finally, the discussed cooling approach was shown to work for many-particle ensembles in the presence of spontaneous emission. Thus, it can be considered as a possible alternative for the standard cooling techniques where the latter cannot be applied or has severe limits.

Acknowledgments

This work was supported by Saint-Petersburg State University (project No. 11.38.640.2013).

References

[1] Carr L D, DeMille D, Krems R V and Ye 2009 *New J. Phys.* **11** 055049
[2] Shuman E S, Barry J F, Glenn D R and DeMille D 2009 *Phys. Rev. Lett.* **103** 223001
[3] Shuman E S, Barry J F and DeMille D 2010 *Nature* **467** 820
[4] Mekhov I B and Ritsch H 2012 *J. Phys.* B **45** 102001
[5] Ritsch H, Domokos P, Brennecke F and Esslinger T 2013 *Rev. Mod. Phys.* **85** 553
[6] Domokos P and Ritsch H 2002 *Phys. Rev. Lett.* **89** 253003
[7] Black A T, Chan H W and Vuletic V 2003 *Phys. Rev. Lett.* **91** 203001
[8] Black A T, Thompson J K and Vuletic V 2005 *J. Phys.* B *At. Mol. Opt. Phys.* **38** S605
[9] Raizen M G, Koga J, Sundaram B, Kishimoto Y, Takuma H and Tajima T 1998 *Phys. Rev.* A **58** 4757
[10] Maunz P, Puppe T, Schuster I, Syassen N, Pinkse P W and Rempe G 2004 *Nature* **428** 6978
[11] Ivanov D, Wallentowitz S and Walmsley I A 2003 *Phys. Rev.* A **67** 061401
[12] Steck D A, Jacobs K, Mabuchi H, Bhattacharya T and Habib S 2004 *Phys. Rev. Lett.* **92** 223004
[13] Averbukh I S and Prior Y 2005 *Phys. Rev. Lett.* **94** 153002
[14] Ivanova T and Ivanov D 2005 *JETP Lett.* **82** 539
[15] Ivanov D A and Ivanova T Y 2014 *J. Phys. B At. Mol. Opt. Phys.* **47** 135303
[16] Grimm R, Weidemüller M and Ovchinnikov Y B 2000 *Adv. At. Mol. Opt. Phys.* **42** 95
[17] Gieseler J, Deutsch B, Quidant R and Novotny L 2012 *Phys. Rev. Lett.* **109** 103603
[18] Asenbaum P, Kuhn S, Nimmrichter S, Sezer U and Arndt M 2013 *Nat. Commun.* **4** 2743
[19] Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge University Press)
[20] Louisell W H 1990 *Quantum Statistical Properties of Radiation* (New York, NY: Wiley)
[21] Collett M J and Gardiner C W 1984 *Phys. Rev.* A **30** 1386
[22] Kloeden P and Platen E 1992 *Numerical Solution of Stochastic Differential Equations* (Berlin: Springer) p 636
[23] Gangl M and Ritsch H 1999 *Phys. Rev.* A **61** 011402