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Letter

Analytical on-shell calculation of low energy higher order scattering

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Abstract

We demonstrate that the use of analytical on-shell methods involving calculation of the discontinuity across the t-channel cut associated with the exchange of a pair of massless particles (photons or gravitons) can be used to evaluate one-loop contributions to electromagnetic and gravitational scattering, with and without polarizability, reproducing via simple algebraic manipulations, results obtained previously, generally using Feynman diagram techniques. In the gravitational case the use of factorization permits a straightforward and algebraic calculation of higher order scattering without consideration of ghost contributions or of triple-graviton couplings, which made previous evaluations considerably more arduous.

Keywords: Compton scattering, gravitational Compton scattering, quantum gravity

1. Introduction

The calculation of scattering amplitudes is a staple of theoretical physics, and recently a number of investigations have been reported which study higher order effects in electromagnetic scattering [1–4], gravitational scattering [5–11] and both [12–14]. The goal of such calculations has typically been to find an effective potential which characterizes these higher order effects. For both electromagnetic and gravitational interactions, the leading potential is, of course, well-known and has the familiar $1/r$ fall-off with distance. The higher order contributions required by quantum mechanics lead to corrections which are shorter range, from local to polynomial fall-off as $1/r^n$ with $n \geqslant 2$. This effective potential is defined to be the Fourier transform of the nonrelativistic

scattering amplitude via[1]

$$V(r) = -\int \frac{\mathrm{d}^3q}{(2\pi)^3} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \mathrm{Amp}(\boldsymbol{q}), \tag{2}$$

where $\boldsymbol{q} = \boldsymbol{p}_i - \boldsymbol{p}_f$ is the three-momentum transfer. Then, for lowest order one-photon or one-graviton exchange, the dominant momentum-transfer dependence arises from the massless propagator $\frac{1}{q^2} \overset{NR}{\longrightarrow} \frac{-1}{\boldsymbol{q}^2} + \mathcal{O}(\boldsymbol{q}^4)$, whose Fourier transform

$$\int \frac{\mathrm{d}^3q}{(2\pi)^3} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \frac{1}{\boldsymbol{q}^2} = \frac{1}{4\pi|\boldsymbol{r}|} \tag{3}$$

yields the well-known $1/r$ dependence. By dimensional analysis, it is clear that shorter-range $1/r^2$ and $1/r^3$ behavior can arise only from *nonanalytic* $1/|\boldsymbol{q}|$ and $\ln \boldsymbol{q}^2$ dependence, which are associated with higher order scattering contributions. Analytic momentum dependence from polynomial contributions in such diagrams leads only to short-distance ($\delta^3(\boldsymbol{r})$ and its derivatives) effects. Thus, if we are seeking the long-range corrections, we need identify only the nonanalytic components of the higher order low energy contributions to the scattering amplitude.

The basic idea behind use of on-shell methods is that the scattering amplitude must satisfy the stricture of unitarity, which requires that its discontinuity across the right-hand cut is given by

$$\mathrm{Disc}\, T_{fi} = \mathrm{i}(T_{fi} - T^{\dagger}{}_{fi}) = -\sum_n T_{fn} T^{\dagger}_{ni}. \tag{4}$$

By requiring that equation (4) be satisfied, we guarantee that the correct nonanalytic structure will be maintained, and below we demonstrate how this program can be carried out in the case of the electromagnetic and gravitational scattering of spinless particles, with and without polarizability effects[2]. We will show how results obtained previously using Feynman diagram techniques can be obtained by *much* simplified analytical on-shell methods. This simplification arises essentially due the interchange of the order of integration and summation. That is, in the conventional Feynman technique, one evaluates separate (four-dimensional) Feynman integrals for each diagram, which are then summed. In the on-shell method, one first sums over the Compton scattering diagrams to obtain helicity amplitudes and *then* performs a (two-dimensional) solid-angle integration. There are a number of reasons why the latter procedure is more efficient. For one, by using the explicitly *gauge-invariant* Compton and gravitational Compton amplitudes, the decomposition into separate *gauge-dependent* diagrams is avoided. Secondly, the various statistical/combinatorial factors are included automatically. Thirdly, because the intermediate states are on-shell, there is no gravitational ghost contribution [16]. Finally, the evaluation of gravitational Compton amplitudes allows the use of factorization, which ameliorates the need to include the triple graviton coupling associated with the graviton pole diagram [17, 18]. The superposition of all these effects allows a relatively simple and highly efficient algebraic calculation of both the electromagnetic and gravitational scattering amplitudes. (One indication of the simplicity

[1] Note that equation (2) follows from the Born approximation for the scattering amplitude

$$\mathrm{Amp}(\boldsymbol{q}) = \langle \boldsymbol{p}_f | \hat{V} | \boldsymbol{p}_i \rangle = \int \mathrm{d}^3 r \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} V(r) \tag{1}$$

and nonrelativistic amplitudes are defined by taking the low energy limit and dividing the covariant forms by the normalizing factor $4E_A E_B \simeq 4m_A m_B$.

[2] It is interesting to note that this method is essentially the one used by Feynman in his seminal paper, wherein he quantized gravity and realized the need for the introduction of ghosts [15].

afforded by this method, in the gravitational case, is that between the seminal 1994 work of Donoghue [5] and the 2003 papers by [9, 10], there were a number of reported Feynman diagram calculations of gravitational scattering which contained errors [6–8].)

2. Electromagnetic scattering

We begin with the case of electromagnetic scattering of spinless particles of mass m_A and m_B respectively. The t-channel Compton amplitude (i.e., the amplitude for two spinless particles of charge e and mass m_A to annihilate into a pair of photons) is well known [19]

$$^{A}\mathrm{Amp}_0^{em} = 2e^2\left(\epsilon_1^*\cdot\epsilon_2^* - \frac{\epsilon_1^*\cdot p_1\,\epsilon_2^*\cdot p_2}{p_1\cdot k_1} - \frac{\epsilon_1^*\cdot p_2\,\epsilon_2^*\cdot p_1}{p_1\cdot k_2}\right). \tag{5}$$

It is convenient to use the helicity formalism [20], where helicity is defined as the projection of the photon spin on its momentum axis. The helicity amplitudes for t-channel spin-0-spin-0 Compton annihilation are found, in the center of mass frame, to have the form [21]

$$\begin{aligned} {}^{A}A_0^{EM}(++) &= {}^{A}A_0^{EM}(--) = 2e^2\left(\frac{m_A^2}{E_A^2 - \boldsymbol{p}_A^2\cos^2\theta_A}\right),\\ {}^{A}A_0^{EM}(+-) &= {}^{A}A_0^{EM}(-+) = 2e^2\left(\frac{\boldsymbol{p}_A^2\sin^2\theta_A}{E_A^2 - \boldsymbol{p}_A^2\cos^2\theta_A}\right), \end{aligned} \tag{6}$$

where m_A, E_A, $\pm\boldsymbol{p}_A$ are the mass, energy, momentum of the spinless particles and θ_A the angle of the outgoing photon with respect to the incoming target particle—$\cos\theta_A = \hat{\boldsymbol{p}}_A\cdot\hat{\boldsymbol{k}}$. It was shown by Feinberg and Sucher that the annihilation amplitudes $A + A' \to \gamma_1 + \gamma_2$ and $\gamma_1 + \gamma_2 \to B + B'$ needed in the unitarity relation, equation (4), can be generated by making an analytic continuation to imaginary momentum $\boldsymbol{p}_i \to \mathrm{i}m_i\xi_i\hat{\boldsymbol{p}}_i$, where $\xi_i^2 = 1 - \frac{t}{4m_i^2}$ with $i = A, B$ and $t = (p_A + p_A')^2$ is the t-channel Mandelstam variable [3]. Then

$$\begin{aligned} {}^{i}A_0^{em}(++) &= {}^{i}A_0^{em}(--) = 2e^2\frac{1+\tau_i^2}{d_i},\\ {}^{i}A_0^{em}(+-) &= {}^{i}A_0^{em}(-+) = 2e^2\frac{1-x_i^2}{d_i}, \end{aligned} \tag{7}$$

where we have defined $\tau_i = \sqrt{t}/2m_i\xi_i$, $x_i = \hat{\boldsymbol{p}}_i\cdot\hat{\boldsymbol{k}}_1$, and $d_i = \tau_i^2 + x_i^2$. Equivalently equation (7) can be represented succinctly via

$${}^{i}A_0^{em}(ab) = 2e^2\mathcal{O}_i^{jk}\epsilon_{1j}^{a*}\epsilon_{2k}^{b*}, \tag{8}$$

where

$$\mathcal{O}_i^{jk} = \frac{1}{d_i}(d_i\delta^{jk} + 2\hat{p}_i^{\,j}\hat{p}_i^{\,k}) \quad i = A, B. \tag{9}$$

Substituting in equation (4), we determine the discontinuity of the scattering amplitude of spinless particles having masses m_A, m_B across the t-channel two-photon cut in the CM frame[3]

[3] Note that we have divided the scattering amplitude by the normalizing factor $4m_Am_B$ since it will be used in the nonrelativistic limit.

$$\begin{aligned}\text{Disc Amp}_2^{em}(q) &= -\frac{\mathrm{i}}{2!}\frac{(2e^2)^2}{4m_Am_B}\int\frac{\mathrm{d}^3k_1}{(2\pi)^3 2k_{10}}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2k_{20}}(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\\ &\quad\times\sum_{a,b=1}^{2}[\mathcal{O}_A^{ij}\epsilon_{1i}^{a*}\epsilon_{2j}^{b*}\epsilon_{1k}^{a}\epsilon_{2\ell}^{b}\mathcal{O}_B^{k\ell *}]\\ &= -\mathrm{i}\frac{e^4}{16\pi m_Am_B}\left\langle\sum_{i,j,k,\ell=1}^{3}\mathcal{O}_A^{ij}\delta_{ik}^T\delta_{j\ell}^T\mathcal{O}_B^{k\ell *}\right\rangle,\end{aligned} \tag{10}$$

where

$$\delta_{ik}^T=\sum_{a=1}^{2}\epsilon_i^{a*}\epsilon_k^a=\delta_{ik}-\hat{k}_i\hat{k}_k \tag{11}$$

represents the sum over photon polarizations and

$$\langle G\rangle\equiv\int\frac{\mathrm{d}\Omega_{\hat{\boldsymbol{k}}}}{4\pi}G$$

defines the solid-angle average. Performing the indicated polarization sums, we find

$$\begin{aligned}\left\langle\sum_{i,j,k,\ell=1}^{3}\mathcal{O}_A^{ij}\delta_{ik}^T\delta_{j\ell}^T\mathcal{O}_B^{k\ell *}\right\rangle &= \left\langle\frac{1}{d_Ad_B}(4(y-x_Ax_B)^2-2(1-x_A^2)(1-x_B^2)\right.\\ &\left.+2(1+\tau_A^2)(1+\tau_B^2))\right\rangle\overset{t<<m_A^2,m_B^2}{\longrightarrow}\left\langle\frac{1}{d_Ad_B}(4(y-x_Ax_B)^2+2x_A^2+2x_B^2-2x_A^2x_B^2)\right\rangle,\end{aligned} \tag{12}$$

where

$$y(s,t)=\hat{\boldsymbol{p}}_A\cdot\hat{\boldsymbol{p}}_B=\frac{2s+t-2m_A^2-2m_B^2}{4m_A\xi_Am_B\xi_B}\overset{s\to s_0}{\longrightarrow}1+\mathcal{O}(t). \tag{13}$$

Characterizes the angle between incoming and outgoing spinless particles. Near threshold —$s_0=(m_A+m_B)^2$—$y(s_0,t)=1+\mathcal{O}(t)$ and

$$\begin{aligned}\text{Disc Amp}_2^{em}(q) &\simeq -\mathrm{i}\frac{e^4}{8\pi m_Am_B}\left\langle\frac{1}{d_Ad_B}(2y^2-4yx_Ax_B+x_A^2+x_B^2+x_A^2x_B^2)\right\rangle\\ &\overset{y\to1}{\longrightarrow}-\mathrm{i}\frac{e^4}{8\pi m_Am_B}\left\langle\frac{1}{d_Ad_B}(2-4x_Ax_B+x_A^2+x_B^2+x_A^2x_B^2)\right\rangle.\end{aligned} \tag{14}$$

Defining

$$I_{nm}\equiv\left\langle\frac{x_A^nx_B^m}{d_Ad_B}\right\rangle, \tag{15}$$

the required angular integrals I_{00}, I_{11} have been given by Feinberg and Sucher [22], and higher order forms can be found by use of the identities $x_i^2=d_i-\tau_i^2$, $i=A,B$, yielding

$$\begin{aligned}\text{Disc Amp}_2^{em}(q) &\simeq -\mathrm{i}\frac{e^4}{8\pi m_Am_B}[2I_{00}-4I_{11}+I_{20}+I_{02}+I_{22}]\\ &= -\mathrm{i}\frac{e^4}{8\pi m_Am_B}\left[\frac{\pi(m_A+m_B)}{\sqrt{-t}}+\frac{7}{3}+\mathrm{i}\frac{4\pi m_Am_Bm_r}{p_0t}+\ldots\right],\end{aligned} \tag{16}$$

where $p_0 = \sqrt{\frac{m_r(s-s_0)}{m_A+m_B}}$ is the center of mass momentum for the spinless scattering process and $m_r = m_A m_B/(m_A + m_B)$ is the reduced mass. Since

$$\text{Disc}\left[\log(-t), \sqrt{\frac{1}{-t}}\right] = \left[2\pi\text{i}, -\text{i}\frac{2\pi^2}{\sqrt{-t}}\right] \tag{17}$$

the scattering amplitude is[4]

$$\text{Amp}_2^{em}(q) = -\frac{e^4}{16\pi^2 m_A m_B}\left[\frac{7}{3}L - S(m_A + m_B) + 4\pi i\frac{m_r m_A m_B}{p_0 t}L + \ldots\right], \tag{18}$$

where we have defined $L = \log(-t)$ and $S = \pi^2/\sqrt{-t}$. The imaginary component of equation (18) represents the Coulomb phase, or equivalently the contribution of the second Born approximation, which must be subtracted in order to define a proper higher-order potential. Using [23]

$$\begin{aligned} B_2^{em}(q) &= \text{i}\int\frac{\text{d}^3\ell}{(2\pi)^3}\frac{e^2}{|\boldsymbol{p}_f - \boldsymbol{\ell}|^2 + \lambda^2}\frac{\text{i}}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + \text{i}\epsilon}\frac{e^2}{|\boldsymbol{\ell} - \boldsymbol{p}_i|^2 + \lambda^2} \\ &\xrightarrow{\lambda\to 0} -\text{i}\frac{e^4}{4\pi}\frac{m_r}{p_0}\frac{\log(-t)}{t} \end{aligned} \tag{19}$$

what remains is the higher order electromagnetic amplitude we are seeking, leading to the effective potential

$$\begin{aligned} V_2^{em}(r) &= -\int\frac{\text{d}^3q}{(2\pi)^3}\text{e}^{-\text{i}\boldsymbol{q}\cdot\boldsymbol{r}}(\text{Amp}_2^{em}(q) - B_2^{em}(q)) \\ &= -\frac{\alpha_{em}^2(m_A + m_B)}{2r^2} - \frac{7\alpha_{em}^2\hbar}{6\pi r^3}, \end{aligned} \tag{20}$$

where $\alpha_{em} = \frac{e^2}{4\pi}$ is the fine structure constant, in agreement with the form calculated by Feynman diagram methods [4].

One can also calculate the interaction of a charged and a neutral polarizable spinless system. For the neutral system we use the Hamiltonian

$$\begin{aligned} H_{\text{eff}} &= -\frac{1}{2}(4\pi\alpha_{\text{E}}\boldsymbol{E}^2 + 4\pi\beta_{\text{M}}\boldsymbol{H}^2) \\ &= -\frac{4\pi\alpha_{\text{E}}}{2m_A^2}p_1^{\alpha}F_{\alpha\beta}F^{\beta\gamma}p_{1\gamma} - \frac{4\pi\beta_{\text{M}}}{8m_A^2}\varepsilon^{\alpha\beta\gamma\delta}F_{\beta\gamma}p_{1\delta}\varepsilon_{\alpha}{}^{\rho\sigma\lambda}F_{\rho\sigma}p_{1\lambda}, \end{aligned} \tag{21}$$

where α_{E}, β_{M} are the electric, magnetic dipole polarizabilities and $F_{\mu\nu}$ is the electromagnetic field tensor, which yields the contact helicity amplitude for emission of a photon pair

$$N_A^{ab} = \pi t\mathcal{V}_A^{ij}\epsilon_{1i}^{a*}\epsilon_{2j}^{b*} \tag{22}$$

[4] Note that in making the transition from equation (16) to (18) it may appear that we are assuming the existence of an *unsubtracted* dispersion relation. However, if one or more subtractions were to be employed the result would be that equation (16) is only changed by a polynomial $P(t) = a + bt + \ldots$. Since the polynomial is analytic in t, it would lead only, after Fourier transforming, to a change in the very short-distance form of the effective potential, *not* to the long-range power law fall-off that arises from nonanalytic terms.

with

$$\mathcal{V}_A^{ij} = (\alpha_{\rm E}^A x_A^2 - \beta_{\rm M}^A(2 - x_A^2))\delta^{ij} + 2(\alpha_{\rm E}^A + \beta_{\rm M}^A)\hat{p}_A^i\hat{p}_A^j. \tag{23}$$

Using the angular averaged integral $J_{00}^B = \left\langle \frac{1}{d_B} \right\rangle = \left(\frac{\pi}{2} - \tan^{-1}\tau_B\right)/\tau_B$ [3] and higher order forms $J_{nm}^B = \left\langle \frac{x_A^n x_B^m}{d_B} \right\rangle$ generated via the use of the substitutions $x_i^2 = d_i - \tau_i^2$, $i = A, B$, the discontinuity for spinless charged-neutral scattering is

$$\begin{aligned}
\text{Disc Amp}_2^{N-em} &= -\frac{\rm i}{2!}\int \frac{{\rm d}^3k_1}{(2\pi)^3 2k_1^0}\frac{{\rm d}^3k_2}{(2\pi)^3 2k_2^0}(2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2) \\
&\quad \times \sum_{a,b=1}^{2}\frac{\pi e^2 t}{m_B}\mathcal{V}_A^{ij}\epsilon_{1i}^{a*}\epsilon_{2j}^{b*}\epsilon_{1k}^{a}\epsilon_{2\ell}^{b}\mathcal{O}_B^{k\ell *} \\
&= \frac{-{\rm i}e^2 t}{16 m_B}\sum_{i,j,k,\ell=1}^{3}\langle[\mathcal{V}_A^{ij}P_{ik}^T P_{j\ell}^T \mathcal{O}_B^{k\ell *}]\rangle \xrightarrow{y\to 1} -2\pi{\rm i}\frac{\alpha_{em}t}{4m_B}[2\alpha_E^A J_{00}^B \\
&\quad + (\alpha_{\rm E}^A + \beta_{\rm M}^A)(J_{02}^B + J_{20}^B - 4J_{11}^B + J_{22}^B)] \\
&= -2\pi{\rm i}\frac{\alpha_{em}t}{4m_B}\left[\left(\frac{2\pi m_B}{\sqrt{-t}} - \frac{11}{3}\right)\alpha_{\rm E}^A - \frac{5}{3}\beta_{\rm M}^A\right].
\end{aligned} \tag{24}$$

The scattering amplitude is then

$$\text{Amp}_2^{N-em}(q) = \alpha_{em}\alpha_{\rm E}^A\frac{\pi^2\sqrt{-t}}{2} + \frac{1}{3}\alpha_{em}(11\alpha_{\rm E}^A + 5\beta_{\rm M}^A)t\log - t \tag{25}$$

and the effective potential describing interaction of a charged and neutral system

$$V_2^{N-em}(r) = -\int\frac{{\rm d}^3q}{(2\pi)^3}{\rm e}^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{r}}\text{Amp}_2^{N-em}(q) = -\frac{\alpha_{em}\alpha_{\rm E}^A}{2r^4} + \alpha_{em}\frac{(11\alpha_{\rm E}^A + 5\beta_{\rm M}^A)\hbar}{4\pi m_B r^5} \tag{26}$$

in agreement with well-known forms [24, 25].

Finally, we can examine the interaction of two spinless systems, both of which are characterized by polarizabilities [26]. Defining the angular averaged quantities $K_{nm} = \langle x_A^n x_B^m\rangle$, we have

$$\begin{aligned}
\text{Disc Amp}_2^{NN-em}(q) &= -\frac{\rm i}{2!}\int\frac{{\rm d}^3k_1}{(2\pi)^3 2k_1^0}\frac{{\rm d}^3k_2}{(2\pi)^3 2k_2^0}(2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2) \\
&\quad \times \sum_{a,b=1}^{2}\pi^2 t^2\mathcal{V}_A^{ij}\epsilon_{1i}^{a*}\epsilon_{2j}^{b*}\epsilon_{1k}^{a}\epsilon_{2\ell}^{b}\mathcal{V}_B^{k\ell *} = -{\rm i}\frac{\pi t^2}{16}\sum_{i,j,k,\ell=1}^{3}\langle[\mathcal{V}_A^{ij}P_{ik}^T P_{j\ell}^T\mathcal{V}_B^{k\ell *}] \\
&= -{\rm i}2\pi\frac{t^2}{16}[(\alpha_{\rm E}^A\alpha_{\rm E}^B + \beta_{\rm M}^A\beta_{\rm M}^B)(2K_{00} + K_{20} + K_{02} - 4K_{11} + K_{22}) \\
&\quad + (\alpha_{\rm E}^A\beta_{\rm M}^B + \alpha_{\rm E}^B\beta_{\rm M}^A)(K_{20} + K_{02} - 4K_{11} + K_{22})] \\
&= -{\rm i}2\pi\frac{t^2}{16}\left[(\alpha_{\rm E}^A\alpha_{\rm E}^B + \beta_{\rm M}^A\beta_{\rm M}^B)\frac{23}{15} - (\alpha_{\rm E}^A\beta_{\rm M}^B + \alpha_{\rm E}^B\beta_{\rm M}^A)\frac{7}{15}\right]
\end{aligned} \tag{27}$$

yielding the scattering amplitude

$$\text{Amp}_2^{NN-em}(q) = \frac{t^2 L}{16}\left[(\alpha_{\rm E}^A\alpha_{\rm E}^B + \beta_{\rm M}^A\beta_{\rm M}^B)\frac{23}{15} - (\alpha_{\rm E}^A\beta_{\rm M}^B + \alpha_{\rm E}^B\beta_{\rm M}^A)\frac{7}{15}\right] \tag{28}$$

and the effective potential

$$\begin{aligned} V_2^{NN-em}(r) &= -\int \frac{\mathrm{d}^3q}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\mathrm{Amp}_2^{NN-em}(q) \\ &= \frac{-23(\alpha_{\mathrm{E}}^A\alpha_{\mathrm{E}}^B+\beta_{\mathrm{M}}^A\beta_{\mathrm{M}}^B)+7(\alpha_{\mathrm{E}}^A\beta_{\mathrm{M}}^B+\alpha_{\mathrm{E}}^B\beta_{\mathrm{M}}^A)}{4\pi r^7}, \end{aligned} \tag{29}$$

which has the familiar Casimir–Polder form [27, 28].

3. Gravitational scattering

These electromagnetic results were first obtained using related methods by Feinberg and Sucher [3, 25]. The real power and simplicity of the on-shell methods, however, is found in the *gravitational* scattering case, where what is needed is the amplitude for the annihilation of a spinless pair of particles of mass m_A and m_B connected by a two-*graviton* intermediate state. A major simplification in this regard is provided by factorization, which avers that the tree level gravitational Compton scattering amplitude is given in terms of the product of ordinary tree level Compton scattering amplitudes multiplied by a simple kinematic factor [17, 18]. That is, defining

$$F = \frac{p_i\cdot k_i p_i\cdot k_f}{k_i\cdot k_f}, \tag{30}$$

we have the remarkable identity for the gravitational Compton amplitude ${}^A\mathrm{Amp}_0^{\mathrm{grav}}$

$$\begin{aligned} {}^A\mathrm{Amp}_0^{\mathrm{grav}} &= \frac{\kappa^2}{8e^4}F\,({}^A\mathrm{Amp}_0^{em})^2 \\ &= \frac{\kappa^2}{2}\left(\frac{p_i\cdot k_i p_f\cdot k_f}{k_i\cdot k_f}\right)\left(\frac{\epsilon_i\cdot p_i\,\epsilon_f^*\cdot p_f}{p_i\cdot k_i}-\frac{\epsilon_i\cdot p_f\,\epsilon_f^*\cdot p_i}{p_i\cdot k_f}-\epsilon_i\cdot\epsilon_f^*\right) \\ &\quad\times\left(\frac{\epsilon_i\cdot p_i\,\epsilon_f^*\cdot p_f}{p_i\cdot k_i}-\frac{\epsilon_i\cdot p_f\,\epsilon_f^*\cdot p_i}{p_i\cdot k_f}-\epsilon_i\cdot\epsilon_f^*\right). \end{aligned} \tag{31}$$

Using equation (31), the two-graviton helicity amplitudes can be written in the factorized form

$$\begin{aligned} {}^AB_0^{\mathrm{grav}}(ij) &= \frac{\kappa^2 m_A^2\xi_A^2 d_A}{4}\mathcal{O}_A^{rs}\epsilon_r^{*i}\epsilon_s^{*j}\mathcal{O}_A^{uv}\epsilon_u^{*i}\epsilon_v^{*j}, \\ {}^AB_0^{\mathrm{grav}}(ij) &= \frac{\kappa^2 m_B^2\xi_B^2 d_B}{4}\epsilon_r^i\epsilon_s^j\mathcal{O}_B^{rs}\epsilon_u^i\epsilon_v^j\mathcal{O}_B^{uv}. \end{aligned} \tag{32}$$

We can then write the unitarity relation for the gravitational scattering of spinless particles of mass m_A, m_B as

$$\begin{aligned}\text{Disc Amp}_2^{\text{grav}}(q) &= -\frac{\mathrm{i}}{2!}\frac{\kappa^4 m_A^2\xi_A^2 m_B^2\xi_B^2}{64 m_A m_B}\int\frac{\mathrm{d}^3k_1}{(2\pi)^3 2k_{10}}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2k_{20}}\\ &\times(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)d_A d_B\sum_{r,s=1}^{2}[\mathcal{O}_A^{ij}\mathcal{O}_A^{k\ell}\epsilon_{1i}^{r*}\epsilon_{2j}^{s*}\epsilon_{1k}^{r*}\epsilon_{2\ell}^{s*}\epsilon_{1a}^{r}\epsilon_{2b}^{s}\epsilon_{1c}^{r}\epsilon_{2d}^{s}\mathcal{O}_B^{ab*}\mathcal{O}_B^{cd*}]\\ &= -\mathrm{i}\frac{\kappa^4 m_A^2\xi_A^2 m_B^2\xi_B^2}{1024\pi m_A m_B}\left\langle d_A d_B\sum_{i,j,k,\ell=1}^{3}\sum_{a,b,c,d=1}^{3}\mathcal{O}_A^{ij}\mathcal{O}_A^{k\ell}P^G_{ik;ac}P^G_{j\ell;bd}\mathcal{O}_B^{ab*}\mathcal{O}_B^{cd*}\right\rangle,\end{aligned} \tag{33}$$

where the sum over graviton polarizations is

$$P^G_{ik;ac}=\sum_{r=1}^{2}\epsilon_i^{r*}\epsilon_k^{r*}\epsilon_a^{r}\epsilon_c^{r}=\frac{1}{2}[\delta^T_{ia}\delta^T_{kc}+\delta^T_{ic}\delta^T_{ka}-\delta^T_{ik}\delta^T_{ac}]. \tag{34}$$

Performing the polarization sum, we find

$$\begin{aligned}\langle[d_A d_B\mathcal{O}_A^{ij}\mathcal{O}_A^{k\ell}P^G_{ik;ac}P^G_{j\ell;bd}\mathcal{O}_B^{ab*}\mathcal{O}_B^{cd*}]\rangle &= \left\langle\frac{1}{d_A d_B}[4(2(y-x_A x_B)^2\right.\\ &\left.-(1-x_A^2)(1-x_B^2))^2-2(1-x_A^2)^2(1-x_B^2)^2+2(1+\tau_A^2)^2(1+\tau_B^2)^2]\right\rangle\\ &\xrightarrow{t<<m_A^2,m_B^2}\left\langle\frac{2}{d_A d_B}[8(y-x_A x_B)^4-8(y-x_A x_B)^2(1-x_A^2)(1-x_B)^2\right.\\ &\left.+(1-x_A^2)^2(1-x_B^2)^2+1]\right\rangle.\end{aligned} \tag{35}$$

Using the angular-averaged forms I_{nm} defined earlier, we determine then

$$\begin{aligned}\text{Disc Amp}_2^{\text{grav}}(q) &\simeq -\mathrm{i}\frac{\kappa^4 m_A\xi_A^2 m_B\xi_B^2}{512\pi}\left\langle\frac{1}{d_A d_B}[8(y-x_A x_B)^4\right.\\ &\left.-8(y-x_A x_B)^2(1-x_A^2)(1-x_B)^2+(1-x_A^2)^2(1-x_B^2)^2+1]\right\rangle\\ &= -\mathrm{i}\frac{\kappa^4 m_A\xi_A^2 m_B\xi_B^2}{512\pi}[I_{44}+6I_{42}+6I_{24}-16I_{33}-16I_{13}-16I_{31}\\ &+I_{40}+I_{04}+36I_{22}+6I_{20}+6I_{02}-16I_{11}+2I_{00}]\\ &= -\mathrm{i}\frac{\kappa^4 m_A m_B}{512\pi}\left[\frac{41}{5}+6\frac{\pi(m_A+m_B)}{\sqrt{-t}}+4\pi\mathrm{i}\frac{m_A m_B m_r}{p_0 t}+\ldots\right]\end{aligned} \tag{36}$$

so that the gravitational scattering amplitude is

$$\text{Amp}_2^{\text{grav}}(q)=-\frac{\kappa^4 m_A m_B}{1024\pi^2}\left[\frac{41}{5}L-6S(m_A+m_B)+4\pi\mathrm{i}\frac{m_A m_B m_r}{p_0 t}L+\ldots\right]. \tag{37}$$

As in the electromagnetic case, the imaginary piece is associated with the gravitational scattering phase, or equivalently the Born iteration, which must be subtracted in order to generate a properly defined second order potential

$$B_2^{\rm grav}(q) = {\rm i}\int \frac{{\rm d}^3\ell}{(2\pi)^3}\frac{\frac{1}{8}\kappa^2 m_A^2}{|\boldsymbol{p}_f-\boldsymbol{\ell}|^2+\lambda^2}\frac{\rm i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+{\rm i}\epsilon}\frac{\frac{1}{8}\kappa^2 m_B^2}{|\boldsymbol{\ell}-\boldsymbol{p}_i|^2+\lambda^2} \xrightarrow{\lambda\to 0} -{\rm i}\frac{\kappa^4}{256\pi}m_A^2 m_B^2\frac{m_r}{p_0 t}L. \tag{38}$$

The result is the well-defined second order gravitational potential

$$V_2^{\rm grav}(r) = -\int\frac{{\rm d}^3q}{(2\pi)^3}{\rm e}^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{r}}({\rm Amp}_2^{\rm grav}(\boldsymbol{q})-B_2^{\rm grav}(\boldsymbol{q})) = -\frac{3G^2m_Am_B(m_A+m_B)}{r^2}-\frac{41G^2m_Am_B\hbar}{10\pi r^3}, \tag{39}$$

which agrees with the result calculated via Feynman diagram methods by Khriplovich and Kirilin and Bjerrum-Bohr *et al* in [9, 10].

We can also deal with a polarizable gravitational system by use of the effective Hamiltonian

$$H = -\frac{1}{2}\alpha_G^A\sum_{ij}R^2_{0i;0j} = -\frac{\alpha_G^A}{2m_A^4}p_1^\alpha p_1^\gamma R_{\alpha\beta;\gamma\delta}R^{\rho\beta;\sigma\delta}p_{1\rho}p_{1\sigma}, \tag{40}$$

where α_G is the quadrupole polarizability and $R_{\alpha\beta;\gamma\delta}$ is the Riemann curvature tensor, which leads to the contact helicity amplitude for emission of a graviton pair

$$S_A^{ab} = t^2\mathcal{U}^{ij;kl}\epsilon_{1i}^{a*}\epsilon_{1k}^{a*}\epsilon_{2j}^{b*}\epsilon_{2\ell}^{b*} \tag{41}$$

with

$$\mathcal{U}_A^{ij;k\ell} = \frac{\alpha_G^A}{64}d_A^2\mathcal{O}_A^{ij}\mathcal{O}_A^{k\ell}. \tag{42}$$

For the case of a point mass m_B interacting with a polarizable mass m_A, we have then

$$\begin{aligned}
&{\rm Disc}\,{\rm Amp}_2^{N-{\rm grav}}\\
&= -\frac{\rm i}{2!}\int\frac{{\rm d}^3k_1}{(2\pi)^3 2k_1^0}\frac{{\rm d}^3k_2}{(2\pi)^3 2k_2^0}(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\\
&\quad\times\sum_{a,b=1}^{2}\frac{t^2\kappa^2m_B^2\xi_B^2 d_B}{8m_B}\mathcal{U}_A^{ij;k\ell}\epsilon_{1i}^{a*}\epsilon_{1k}^{a*}\epsilon_{2j}^{b*}\epsilon_{2\ell}^{b*}\epsilon_{1u}^{a}\epsilon_{1v}^{a}\epsilon_{2r}^{b}\epsilon_{2s}^{b}\mathcal{O}_B^{ur*}\mathcal{O}_B^{vs*}\\
&= \frac{-{\rm i}\kappa^2m_B\xi_B^2t^2}{128}\sum_{i,j,k,\ell=1}^{3}\sum_{u,v,r,s=1}^{3}\langle[\mathcal{U}_A^{ij;k\ell}P^T_{ik;uv}P^T_{j\ell;rs}\mathcal{O}_B^{ur*}\mathcal{O}_B^{vs*}]d_B\rangle\\
&\xrightarrow{y\to 1}\frac{\kappa^2m_B\alpha_G^At^2}{4096\pi}[J_{44}^B+6J_{42}^B+6J_{24}^B-16J_{33}^B-16J_{13}^B-16J_{31}^B+J_{40}^B+J_{04}^B\\
&\quad+36J_{22}^B+6J_{20}^B+6J_{02}^B-16J_{11}^B+2J_{00}^B] = \frac{\kappa^2m_B\alpha_G^At^2}{4096\pi}\left[\frac{2\pi m_B}{\sqrt{-t}}-\frac{163}{35}\right]
\end{aligned} \tag{43}$$

so

$$\mathrm{Amp}_2^{N-\mathrm{grav}} = -\frac{Gm_B\alpha_G^A}{256\pi}\left(2m_BSt^2-\frac{163}{35}t^2L\right). \tag{44}$$

Setting $t = q^2$ and taking the Fourier transform, we find the effective potential

$$V_2^{N-\mathrm{grav}}(r) = -\int \frac{\mathrm{d}^3q}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\mathrm{Amp}_2^{N-\mathrm{grav}}(\boldsymbol{q}) = -\frac{3Gm_B^2\alpha_G^A}{32\pi}\left(\frac{1}{r^6} + \frac{163}{7}\frac{\hbar}{\pi m_B r^7}\right), \tag{45}$$

which is, we believe, a new result.

Finally, in the case of the long-range interaction of a pair of polarizable systems, we have

$$\begin{aligned}
&\mathrm{Disc}\,\mathrm{Amp}_2^{NN-\mathrm{grav}}(q)\\
&= -\frac{\mathrm{i}}{2!}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2k_1^0}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2k_2^0}(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\\
&\quad\times\sum_{a,b=1}^{2} t^4\mathcal{U}_A^{ij;k\ell}\epsilon_{1i}^{a*}\epsilon_{1k}^{a*}\epsilon_{2j}^{b*}\epsilon_{2\ell}^{b*}\epsilon_{1u}^{a}\epsilon_{1v}^{a}\epsilon_{2r}^{b}\epsilon_{2s}^{b}\mathcal{U}_B^{uv;rs}\\
&= \frac{t^4}{16}\sum_{i,j,k,\ell=1}^{3}\sum_{u,v,r,s=1}^{3}\langle\mathcal{U}_A^{ik;uv}P_{ik;uv}^{T}P_{j\ell;rs}^{T}\mathcal{U}_B^{uv;rs}\rangle\\
&\quad= -\mathrm{i}\frac{\alpha_G^A\alpha_G^B t^4}{32768\pi}[K_{44}+6K_{42}+6K_{24}-16K_{33}-16K_{13}-16K_{31}+K_{40}+K_{04}\\
&\quad+36K_{22}+6K_{20}+6K_{02}-16K_{11}+2K_{00}] = -\mathrm{i}\frac{\alpha_G^A\alpha_G^B t^4}{32768\pi}\frac{443}{315}
\end{aligned} \tag{46}$$

so

$$\mathrm{Amp}_2^{NN-\mathrm{grav}}(q) = \frac{\alpha_G^A\alpha_G^B t^4}{32768\pi^2}\frac{443}{630}L. \tag{47}$$

Taking the Fourier transform, we find the effective potential

$$V_2^{NN-\mathrm{grav}}(r) = -\int \frac{\mathrm{d}^3q}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\mathrm{Amp}_2^{NN-\mathrm{grav}}(q) = -\frac{3987}{1024}\frac{\alpha_G^A\alpha_G^B}{\pi^3 r^{11}} \tag{48}$$

which agrees precisely with the retarded form given in [29, 30], when we take into account the difference in the definition of polarizability used in our paper (α_G) and theirs (α_{1S})—

$$\alpha_G^A \equiv 16\pi G\alpha_{1S} \quad\text{and}\quad \alpha_G^B \equiv 16\pi G\alpha_{2S}. \tag{49}$$

These methods are also easily adapted to the case that one of the scattering particles is massless, as considered in a recent paper on the bending of light as it passes the rim of the Sun [31]. An important difference is that we must use $m_A^2\xi_A^2 \to -\frac{t}{4}$, $\tau_A \to i$ and $y \to i|y|$ with $|y| = \frac{s+\frac{t}{2}-m_B^2}{m_B\sqrt{-t}}$. Writing $s - m_B^2 = 2m_B E$, where E is the incident energy of the massless particle in the laboratory frame, we work in the small angle scattering approximation $E \gg \sqrt{-t}$ so that $|y| \gg 1$, in which case, using appropriately modified values of $I_{nm} \to I_{nm}^{(0)}$[5],

[5] Note that here we have divided by the normalizing factor $4Em_B$.

$$\begin{aligned}\text{Disc Amp}_2^{0-\text{grav}}(q) \simeq & -\text{i}\frac{1}{64\pi}\kappa^4 t m_B\left[4\frac{E^3}{t^2}I_{00}^{(0)}+\text{i}8\frac{E^2}{t\sqrt{-t}}I_{11}^{(0)}-3\frac{E}{t}J_{00}^{A(0)}\right.\\ & \left.-\frac{15}{16}\frac{E}{t}J_{00}^{B}-\frac{3}{80}\frac{E}{t}\right]\\ = & -\text{i}\frac{\kappa^4}{256\pi m_B E}\left(2m_B E+\frac{t}{2}\right)^2\\ & \times\left[\frac{\left(2m_B E+\frac{t}{2}\right)^2}{t}\left(\frac{m_B}{2E}\ln\left(\frac{2E}{m_B}\right)-\frac{m_B^2}{2m_B E+t}\ln\left(\frac{-m_B^2}{2m_B E+t}\right)\right)\right.\\ & -2\left(2m_B E+\frac{t}{2}\right)\left(\frac{m_B}{2E}\ln\left(\frac{2E}{m_B}\right)+\frac{m_B}{2m_B E+t}\ln\left(\frac{-m_B^2}{2m_B E+t}\right)\right)\\ & \left.+\frac{3}{2}L-\frac{15}{16}\left(\frac{\pi m_B}{\sqrt{-t}}-1\right)-\frac{3}{80}\right]\end{aligned} \tag{50}$$

so

$$\begin{aligned}\text{Amp}_2^{0\text{-grav}}(q) \simeq & -\frac{\kappa^4 m_B E}{512\pi^2}\left[\frac{L}{t}\left[2m_B E\ln\left(\frac{2E}{m_B}\right)-(2m_B E+t)\ln\left(-\frac{2m_B E+t}{m_B^2}\right)\right]\right.\\ & \left.+3L^2+\frac{15}{4}(L+Sm_B)-\frac{3}{20}L\right].\end{aligned} \tag{51}$$

As $t\to 0$ the sum of the two terms in the top line of equation (51) becomes imaginary, corresponding to a scattering phase,

$$B_2^{0-\text{grav}}(q)=-\text{i}\frac{\kappa^4 m_B E L}{512\pi t} \tag{52}$$

which must, as previously, be subtracted[6]. The resulting potential, written in terms of the laboratory frame energy of the massless particle E, is

$$\begin{aligned}V_2^{0-\text{grav}}(r) = & -\int\frac{\text{d}^3q}{(2\pi)^3}\text{e}^{-\text{i}\boldsymbol{q}\cdot\boldsymbol{r}}(\text{Amp}_2^{0-\text{grav}}(q)-B_2^{0-\text{grav}}(q))\\ = & \frac{15}{4}\frac{G^2 m_B^2 E}{r^2}-\frac{\left(15-\frac{3}{5}\right)}{4\pi}\frac{G m_B E\hbar}{r^3}-\frac{12G^2 m_B E\hbar}{\pi r^3}\ln\frac{r}{r_0}\end{aligned} \tag{53}$$

and agrees with the form calculated in [31].

4. Conclusion

We have seen above how on-shell techniques can be used in order to treat higher order contributions in an entire range of spinless scattering reactions, including the electromagnetic scattering of charged systems, of charged and neutral (polarizable) systems, and of two neutral systems. Similarly in the case of gravitational interactions the interactions of two masses, of a mass and a polarizable system, and two polarizable systems can be dealt with. In

[6] Using this condition, we also find that the BCJ relation is satisfied, which serves as an additional check on the calculation [32].

each case the calculation is found to agree with previously found forms but is accomplished via an algebraic on-shell method which is considerably simpler to use than the corresponding Feynman diagram procedure. This simplification arises due to the reordering of the Feynman integration and diagram summations and, in the gravitational case, to the use of factorization, which means that the required helicity amplitudes are given simply in terms of the product of electromagnetic amplitudes. An additional bonus is the feature that, since we are on-shell, there is no need to include ghost contributions. We also found that this method could be adapted to the case that one of the scattering systems becomes massless. The result is a highly efficient method to evaluate higher order electromagnetic and gravitational scattering amplitudes. (Note that similar methods have been used by Bjerrum-Bohr *et al* [33, 34]. The basic difference between their work and that described above is that these authors use covariant evaluation and then expand to yield the low energy forms, which requires somewhat more work than the technique described above.) Work is underway to extend our results to the case that both scattering systems are massless and to the situation that one or both of the scattering systems carries spin.

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