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Hamiltonian formulation of the linearized hypersurface-orthogonal Einstein-aether theory

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Abstract. We perform the Hamiltonian formulation of the Einstein-aether theory subject to the condition of hypersurface-orthogonality on the aether vector. Our main aim is to obtain the bulk Hamiltonian together with its constraints and their algebra. We implement a perturbative approach, studying the theory at the linearized level. Under the condition of hypersurface orthogonality the aether vector can be represented by a scalar field. The theory has two first-class constraints, which are associated to the symmetry of general diffeomorphisms over the spacetime. One of the constraints depends on the perturbative aether field, generating linear-order diffeomorphisms on it. We also present the reduced bulk Hamiltonian depending on the physical degrees of freedom. We find conditions on the coupling constants enusring the positiveness of the reduced Hamiltonian and the hyperbolicity of the propagation equations of the independent modes.

1. Introduction

There is a great interest in studying theories that break the Lorentz symmetry, either by quantum motivations or even at the classical level with theories that are modifications of general relativity (GR). One of such classical theories is the Einstein-aether theory (EA-theory) [1]. In esence, this theory modifies GR by coupling to it a dynamical vector field u^μ, called the aether, which is regarded as a fundamental field rather than a matter source. The dynamical aether vector is restricted to be timelike and unitary, $u^\alpha u_\alpha = -1$. On any configuration, the presence of the vector u^μ breaks the local Lorentz symmetry. On the other hand, since the vector u^μ is dynamical and the action is fully written in term of spacetime tensors, the action is manifestly invariant under general diffeomorphisms on the spacetime. The action of the standard theory, which is the one we study here, is of second order in derivatives of u^μ, although higher order couplings can be considered as well.

In the general case there are three independent propagating modes associated to the aether vector. This number can be reduced by imposing the geometrical condition of hypersurface orthogonality on the aether vector u^μ. Locally this condition can be defined by means of a function T whose gradient is everywhere timelike, $\partial_\mu T \partial^\mu T < 0$. Then the condition of hypersurface orthogonality is

$$u_\alpha = \frac{\partial_\alpha T}{(-g^{\mu\nu}\partial_\mu T \partial_\nu T)^{1/2}}. \quad (1)$$

With this condition the normalization constraint on the aether field is satisfied. We call the theory with this condition the restricted EA-theory (other used names are the khronometric theory and the T theory). Intrerestingly, the Lagrangian of the restricted EA-theory is related [2, 3] to the Lagrangian of the nonprojectable Hořava theory [4, 5] truncated up to second order in spatial derivatives.

Regardless of the condition of hypersurface orthogonality, the full Hamiltonian formulation of the EA-theory has not been successfully achieved due to the mathematical complications that arise in the Legendre transformation (boundary terms that determine the energy and other conserved charges for asymptotically flat configurations have been previously studied in Refs. [6, 7]).

In this paper we give an answer to the problem of finding the bulk Hamiltonian of the Einstein-aether theory. Specifically, by means of a perturbative analysis we obtain the bulk Hamiltonian of the linearized restricted EA-theory with its corresponding constraints. This analysis allows us to determine rigorously the dynamics of the true propagating modes of the theory. We also study the relation of the constraints with symmetries.

2. Lagrangian and Hamiltonian formulations

2.1. The perturbative Lagrangian

The action of the (unrestricted) EA-theory, omiting a cosmological-constant term and matter couplings, is [1]:

$$S_{ae} = \int \sqrt{-g}\,(R + \mathcal{L}_{ae})\,d^4x. \tag{2}$$

In this action R is the four-dimensional Ricci scalar and $\mathcal{L}_{ae}$ the Lagrangian density of the aether vector, which is given by

$$\mathcal{L}_{ae} = -\mathcal{M}^{\alpha\beta\lambda\gamma}\nabla_\alpha u_\lambda \nabla_\beta u_\gamma, \tag{3}$$

where $\mathcal{M}^{\alpha\beta\lambda\gamma}$ a matrix defined as

$$\mathcal{M}^{\alpha\beta\lambda\gamma} = c_1 g^{\alpha\beta}g^{\lambda\gamma} + c_2 g^{\alpha\lambda}g^{\beta\gamma} + c_3 g^{\alpha\gamma}g^{\beta\lambda} + c_4 u^\alpha u^\beta g^{\lambda\gamma}. \tag{4}$$

$c_{1,2,3,4}$ are dimensionless coupling constants. The action (2) is manifestly invariant under general diffeomorphism over the spacetime.

The restricted EA-theory is defined by imposing the hypersurface-orthogonality condition (1) at the level of the action (2), i. e., before deriving the equation of motion. We assume this restriction from now on. Thus, the action (2) becomes a functional of $g_{\mu\nu}$ and T.

In order to address the Hamiltonian formulation of the restricted EA-theory, we first decompose the spacetime metric $g_{\mu\nu}$ in the standard Arnowitt-Deser-Misner (ADM) variables γ_{ij}, N and N_i. With these variables the action (2) takes the form

$$S_{ae} = \int N\sqrt{\gamma}\left(K^{ij}K_{ij} - K^2 + {}^{(3)}R - \mathcal{M}^{\alpha\beta\lambda\gamma}\nabla_\alpha u_\lambda \nabla_\beta u_\gamma\right) d^3x dt, \tag{5}$$

where ${}^{(3)}R$ is the spatial Ricci scalar, K_{ij} the extrinsic curvature defined by

$$K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i\right), \tag{6}$$

and $K = \gamma^{ij}K_{ij}$ is the trace of the extrinsic curvature. The dot denotes time derivative, $\dot{\gamma}_{ij} = \partial\gamma_{ij}/\partial t$. It is supposed that the ADM variables are substituted in the aether Lagrangian as well. We comment that there are contributions to the kinetic terms of γ_{ij} coming from the aether sector. Below we shall see this explicitly.

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Now we implement the perturbations. We choose as the background the Minkowski spacetime together with the condition $T = t$, which constitute a solution of the field equations ($T = t$ gives a zero energy-momentum tensor). We introduce the perturbative variables n, n_i, h_{ij} and τ in the following way

$$N = 1 + \epsilon n, \tag{7}$$
$$N^i = \epsilon n_i, \tag{8}$$
$$\gamma_{ij} = \delta_{ij} + \epsilon h_{ij}, \tag{9}$$
$$T = t + \epsilon\tau. \tag{10}$$

For the sake of simplify computations, in the whole perturbative analysis we impose the transverse gauge

$$\partial_i h_{ij} = 0. \tag{11}$$

After introducing the perturbations (7 - 10) in (5) and (1), we obtain the action of quadratic order in perturbations. It is

$$\begin{aligned} S = \frac{1}{\beta}\int dtd^3x \Big[&\kappa_{ij}\kappa_{ij} - \lambda\kappa^2 + \alpha\left(\partial_i\dot{\tau} - \partial_i n\right)^2 + \frac{\beta}{4}\left(h_{ij}\Delta h_{ij} - h\Delta h\right) - \beta n\Delta h \\ &-2\left(1-\beta\right)\kappa_{ij}\partial_{ij}\tau - 2\left(\beta-\lambda\right)\kappa\Delta\tau + (1-\lambda)(\Delta\tau)^2 \Big], \end{aligned} \tag{12}$$

where κ_{ij} is the linear-order perturbation of the extrinsic curvature,

$$\kappa_{ij} = \frac{1}{2}\left(\dot{h}_{ij} - \partial_i n_j - \partial_j n_i\right), \tag{13}$$

$\kappa = \kappa_{kk}$, $h = h_{kk}$ and $\Delta = \partial_{kk}$ is the flat Laplacian. α, β and λ are combinations of the coupling constants defined by

$$\alpha = \frac{c_1 - c_4}{1 - c_1 - c_3}, \quad \beta = \frac{1}{1 - c_1 - c_3}, \quad \lambda = \frac{1 + c_2}{1 - c_1 - c_3}. \tag{14}$$

These combinations, which are the ones the perturbative theory naturally adopts, can be identified with the coupling constants of the $z = 1$ nonprojectable Hořava theory [3]. As we anticipated, there are contributions to the kinetic term of h_{ij} coming from the aether sector; their presence can be recognized by the fact that the coupling constant β^{-1} in front of the action (12) is equal to $1 - c_1 - c_3$.

2.2. The Legendre transformation

From the Lagrangian (12) we obtain the following conjugate momenta

$$\phi_0 = \frac{\partial\mathcal{L}}{\partial\dot{n}} = 0, \quad \phi_i = \frac{\partial\mathcal{L}}{\partial\dot{n}^i} = 0, \tag{15}$$
$$p_\tau = \frac{\partial\mathcal{L}}{\partial\dot{\tau}} = -\frac{2\alpha}{\beta}\Delta\left(\dot{\tau} - n\right), \tag{16}$$
$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = \frac{1}{\beta}\left[\kappa_{ij} - \lambda\kappa\delta_{ij} - (1-\beta)\left(\partial_{ij}\tau\right) - (\beta-\lambda)\,\delta_{ij}\Delta\tau\right]. \tag{17}$$

Equations (15) are primary constraints. Equations (16) and (17) allow us to solve the variables $\dot{\tau}$ and $\dot{h}_{ij}$ in terms of the canonical variables,

$$\dot{\tau} = -\frac{\beta}{2\alpha\Delta}p_\tau + n, \tag{18}$$
$$\kappa_{ij} = \beta\pi^{ij} + \frac{\beta\lambda}{1-3\lambda}\delta_{ij}\pi + (1-\beta)\partial_{ij}\tau + \frac{\beta(1-\lambda)}{1-3\lambda}\delta_{ij}\Delta\tau. \tag{19}$$

Notice that this inversion cannot be performed when the constant λ takes the value $\lambda = 1/3$. In the nonprojectable Hořava theory there is an analog to this particular case. Its Hamiltonian formulation has been studied in Refs. [8, 9]. In that theory two additional second-class constraints emerge and the extra scalar mode is eliminated. Throughout this paper we assume $\lambda \neq 1/3$, hence the inversion (19) is consistent. In addition, Eq. (18) requires $\alpha \neq 0$.

With these formulas we may build the Hamiltonian, which takes the form

$$\begin{aligned} H = \int d^3x \Big[& \frac{1}{\beta}\left(\kappa_{ij}\kappa_{ij} - \lambda\kappa^2\right) - \frac{\beta}{4\alpha} p_\tau \frac{1}{\Delta} p_\tau - \frac{1}{4}(h_{ij}\Delta h_{ij} - h\Delta h) + \frac{1}{\beta}(\lambda - 1)(\Delta\tau)^2 \\ & + n\mathcal{H}^{(1)} + n_i \mathcal{H}^{i(1)} \Big] , \end{aligned} \tag{20}$$

where

$$\mathcal{H}^{(1)} \equiv \Delta h + p_\tau , \tag{21}$$

$$\mathcal{H}^{j(1)} \equiv -2\partial_i \pi^{ij} . \tag{22}$$

We have put the superscript $^{(1)}$ on these objects to remark that they are of linear order in perturbations. In the Hamiltonian (20) it is understood that the variable κ_{ij} must be substituted by its expression (19) in terms of the canonical variables.

We observe that there is a nonlocal term in the Hamiltonian density (20). This nonlocality is a mere effect of using a phase space with nonreduced variables. Below we show that, when the Hamiltonian is reduced to the physical phase space (the reduced Hamiltonian), the resulting Hamiltonian density is completely local.

Our next step is to implement the Dirac procedure for preserving the constraints in time. The preservation in time of the ϕ_0 and ϕ_i constraints leads, respectively, to the secondary constraints

$$\mathcal{H}^{(1)} = 0 , \tag{23}$$

$$\mathcal{H}^{i(1)} = 0 . \tag{24}$$

Subsequently, we impose the preservation in time of these last constraints. We obtain that they are automatically preserved by the Hamiltonian (20),

$$\dot{\mathcal{H}}^{(1)} = \{\mathcal{H}^{(1)}, H\} = 0 . \tag{25}$$

$$\dot{\mathcal{H}}^{i(1)} = \{\mathcal{H}^{i(1)}, H\} = 0 . \tag{26}$$

This closes Dirac's procedure. Therefore, we have that the set of all constraints is given by ϕ_0, ϕ_i, $\mathcal{H}^{(1)}$ and $\mathcal{H}^{i(1)}$, and all these constraints are preserved. The structure of the Hamiltonian (20) allows us to make the usual simplification of this kind of generally-covariant theories: n and n_i can be regarded as Lagrange multipliers, such that the conjugated pairs (n, ϕ_0) and (n_i, ϕ_i) can be dropped out from the phase space.

We comment that the Hamiltonian (20) is composed of two parts: a part that is explicitly the sum of constraints and the other one that is not. On the other hand, this is a theory with general convariance, which implies the freedom of transforming arbitrary the time. For this kind of symmetry one expects the bulk Hamiltonian is a total sum of constraints, such that it vanishes on the constrained phase space. Actually, there is no fundamental contradiction between this and the fact that H in (20) has terms outside the constraints; the discrepancy is only an effect of the perturbative theory. The terms in H that do not explicitly belong to constraints arise as a consequence of truncating the Hamiltonian up to a given order in perturbations. We expect that the nonperturbative Hamiltonian is effectively a sum of constraints, but this nonperturbative Hamiltonian is not known for this theory.

2.3. Symmetries and the reduced Hamiltonian

The two constraints of the theory, $\mathcal{H}^{(1)}$ and $\mathcal{H}^{i(1)}$, are first-class constraints (this can be explicitly checked using the full expressions of the constraints without fixing the gauge (11)). This implies that $\mathcal{H}^{(1)}$ and $\mathcal{H}^{i(1)}$ are generators of gauge symmetries. As in linearized GR, $\mathcal{H}^{i(1)}$ is the generator of purely spatial diffeomorphisms on h_{ij}, that is, it is the momentum constraint. The symmetry associated to this constraint has been already fixed in (11). Interestingly, $\mathcal{H}^{(1)}$ has a double action. It generates transformations on the metric variables associated to time transformations, as does the Hamiltonian constraint of linearized GR; but it also generates redefinitions of τ,

$$\left\{\tau, \int d^3z\, \varphi \mathcal{H}^{(1)}\right\} = \varphi\,. \tag{27}$$

Since φ is an arbitrary test function, this transformation is a total redefinition of τ. This transformation can also be regarded as a diffeomorphism associated to the time coordinate. Indeed, coming back to the nonperturbative field T, we have that it is an scalar under a general diffeomorphism given by $\delta x^\mu = \epsilon^\mu(x^\alpha)$,

$$\delta T = -\epsilon^\mu \partial_\mu T. \tag{28}$$

If we now substitute the perturbative expansion (10) in this transformation rule, at linear order we obtain

$$\delta\tau = -\epsilon^0. \tag{29}$$

Thus, at linear order the variable τ transforms under a diffeomorphism only with the time component (at linear order it is invariant under a diffeomorphism affecting only the spatial directions), and it is totally redefined by the component ϵ^0. As we may see from (27), this transformation on τ is generated by $\mathcal{H}^{(1)}$. The fact that this is a symmetry of the theory means that the dynamics is independent of the particular τ or, in other words, τ is a pure-gauge variable and can be gauged fixed. In the exact theory, this feature can be traced back to the fact that the equation of motion corresponding to the variations of the action with respect to the T field is implied by the other field equations [3].

We are now ready to study the reduced (physical) Hamiltonian, which requires to solve all the constraints and fix the gauge symmetries. The purely spatial diffeomorphisms have been fixed by (11). To fix the diffeomorphisms along the time coordiante we choose the gauge in which τ vanishes,

$$\tau = 0\,. \tag{30}$$

To further advance we decompose the perturbative spatial metric into transverse and longitudinal parts,

$$h_{ij} = h_{ij}^{TT} + \frac{1}{2}\left(\delta_{ij} - \frac{\partial_{ij}}{\Delta}\right) h^T + \partial_j h_i^L + \partial_i h_j^L, \tag{31}$$

where these variables are restricted by $\partial_k h_{kl}^{TT} = h_{kk}^{TT} = 0$. We also decompose π_{ij} in the same way as (31). The transverse gauge (11) eliminates the longitudinal sector h_i^L. By virtue of the constraint (24) the longitudinal sector of π_{ij} is also eliminated. Constraint (23) can be solved for p_τ in favour of h^T,

$$p_\tau = -\Delta h^T. \tag{32}$$

After these considerations, we obtain the reduced Hamiltonian for the true physical degrees of freedom. It takes the form

$$H_{\mathrm{RED}} = \int d^3x \left[\beta \pi_{ij}^{TT}\pi_{ij}^{TT} - \frac{1}{4} h_{ij}^{TT}\Delta h_{ij}^{TT} + \frac{\beta(1-\lambda)}{2\,(1-3\lambda)}(\pi^T)^2 - \frac{2\beta-\alpha}{8\alpha} h^T \Delta h^T\right]. \tag{33}$$

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Of course, the variables h_{ij}^{TT}, π_{ij}^{TT} are still subject to the conditions of transversality and vanishing trace. As we anticipated, the reduced Hamiltonian density is completely local.

In order to ensure the positiveness of the reduced Hamiltonian (33), the following conditions on the coupling constants are required

$$\beta \geq 0, \quad \frac{1-\lambda}{1-3\lambda} \geq 0, \quad \frac{2\beta-\alpha}{\alpha} \geq 0. \tag{34}$$

The second condition can be satisfied only in the ranges $\lambda \in (-\infty, 1/3)$ and $\lambda \in [1, +\infty)$. The third condition can be satisfied only in $\alpha \in (0, 2\beta]$. We recall that we have assumed $\lambda \neq 1/3$ and $\alpha \neq 0$ in all the analysis.

From the reduced Hamiltonian (33) the evolution equations of the several modes can be deduced. From these equations we obtain the second-order equations for h_{ij}^{TT} and h^T,

$$\ddot{h}_{ij}^{TT} - \beta \Delta h_{ij}^{TT} = 0, \tag{35}$$
$$\ddot{h}^{T} - c_s^2 \Delta h^{T} = 0, \tag{36}$$

where

$$c_s = \sqrt{\frac{\beta(2\beta-\alpha)(1-\lambda)}{\alpha(1-3\lambda)}}. \tag{37}$$

The conditions (34) on the coupling constants ensure the hyperbolicity of the Eqs. (35) and (36), hence they are wave equations. The speed of the tensorial modes is $\sqrt{\beta}$ whereas the speed of the scalar mode is c_s.

3. Conclusions

We have successfully performed the Hamiltonian formulation of the linearized Einstein-aether theory with the condition of hypersurface orthogonality on the aether vector, under which the aether vector can be modelled via a scalar field. We have focused the problem on a perturbative approach in order to overcome the difficulties arising in the nonperturbative theory, whose bulk Hamiltonian is still not known. We have found that the theory possesses two first-class constraints associated to the symmetry of general diffeomorphisms over the spacetime. This is in agreement with the fact that the Einstein-aether theory is a theory with general covariance. In particular, one of the constraints generates diffeomorphisms along the time coordinate on the scalar field corresponding to the aether. This symmetry transformation is reinterpreted as this scalar field is a pure-gauge variable. The reduced Hamiltonian is positive if certains bounds on the coupling constants are imposed. These conditions match with the conditions found in Ref. [5] in the context of the nonprojectable Hořava theory. We have found that the restricted EA-theory has three canonical propagating modes, the two transverse-traceless tensorial modes and a scalar mode. They propagate with wave equations with different speeds. This confirms that the condition of hypersurface orthogonality eliminates two of the three modes associated to the aether vector.

It would be interesting to investigate if an iterative process on the perturbative theory can shed some light on the Hamiltonian of the nonperturbative theory. Another interesting application of our results is the comparison with the Hamiltonian formulation of the nonprojectable Hořava theory [10, 11, 12]. The idea is to ellucidate whether both Hamiltonian formulations are physically equivalent. It is of particular interest to perform the analysis with general boundary conditions, such that the relation can be determined for general configurations, not only for asymptotically flat ones.

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