Super-Poynting vector and comoving observers in the Einstein-Rosen spacetime

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Super-Poynting vector and comoving observers in the Einstein-Rosen spacetime

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Abstract. In this work we used the concept of super-energy flux in General Relativity and the monad formalism of Ehlers-Zelmanov to find the comoving observers with the Einstein-Rosen cylindrical gravitational waves and obtained their corresponding 3-velocity of propagation with respect to a family of observers locally at rest. Therefore, we found that the aforementioned waves propagate at subluminal speed in vacuum.

1. Introduction

A number of alternative theories to General Relativity have the feature of subluminal propagation of gravitational waves (e.g. [2,6-7]). Nevertheless in General Relativity it is assumed that gravitational radiation propagates at the speed of light in vacuum [6] for instance this is the case of exact solutions like PP-waves. A special case of these solutions are gravitational plane waves, they are the gravitational analogue of plane electromagnetic waves. It is clear that gravitational waves propagate at the speed of light also in the linearized version of General Relativity. Previous works have shown subluminal propagation of gravitational waves in General Relativity in special cases. Bretón, et al [1] demonstrated that two colliding gravitational plane waves slow down their propagation speed as they approach to the singularity that appear after their collision. Now, if one take into consideration cylindrical gravitational waves (Einstein-Rosen waves) [3,8] the situation is somewhat different. In this case one cans find commoving observers with the waves which of course means subluminal propagation in vacuum. In electromagnetic theory, the Poynting vector is used to measure the flux of energy-momentum density. The corresponding gravitational analog is the super-Poynting vector [4] although the latter is a measure of the flux of a rather different physical quantity, the super-energy. If the super-Poynting vector vanishes in any reference frame, then this frame is at rest or comoving with respect to the gravitational field. Using the tools of the monad method [5,9] we determined the propagation speed of the comoving observers with respect to a family of observers at rest with respect to the symmetry axis in the Einstein-Rosen spacetime.

2. Einstein-Rosen spacetime

The Einstein-Rosen solution found in 1937, describes the propagation in vacuum of cylindrically symmetric monochromatic gravitational waves or pulses, they approach to the axis of symmetry, and then move away of it [3,8]. This solution in cylindrical coordinates is

$$ds^2 = e^{2(K-U)}(dt^2 - d\rho^2) - e^{2U} dz^2 - e^{-2U} \rho^2 d\varphi^2 , \tag{1}$$

where K and U are functions of ρ and t alone, describes the propagation of cylindrical gravitational waves in vacuum. This metric satisfies the vacuum field equations

$$U'' + \rho^{-1} U' - \ddot{U} = 0 , \tag{2}$$

$$K' = \rho(U'^2 + \dot{U}^2), \quad \dot{K} = 2\rho U'\dot{U}. \tag{3}$$

The function U can be in principle any solution to the cylindrical wave equation (2) and K can be obtained by simple integration using equations (3). Two special cases are of physical importance; the first one is that of a monochromatic wave propagating in vacuum, the metric functions are

$$U_m = AJ_0(\omega\rho)\cos(\omega t) + AY_0(\omega\rho)\sin(\omega t) \tag{4}$$

$$\begin{aligned} K_m = \frac{1}{2}A^2\omega\rho\{ & J_0(\omega\rho)J_0'(\omega\rho) + Y_0(\omega\rho)Y_0'(\omega\rho) + \\ & + \omega\rho[(J_0(\omega\rho))^2 + (J_0'(\omega\rho))^2 + (Y_0(\omega\rho))^2 + (Y_0'(\omega\rho))^2] + \\ & + [J_0(\omega\rho)J_0'(\omega\rho) - Y_0(\omega\rho)Y_0'(\omega\rho)]\cos(2\omega t) + \\ & + [J_0(\omega\rho)Y_0'(\omega\rho) + Y_0(\omega\rho)J_0'(\omega\rho)]\sin(2\omega t)\} - \frac{2}{\pi}A^2\omega t, \end{aligned} \tag{5}$$

where $\mathrm{J}_0(\omega\rho)$ and $Y_0(\omega\rho)$ are Bessel functions of first and second kinds respectively, A and ω are the wave amplitude and frequency. The second case is known as the Weber-Wheeler-Bonnor pulse, the metric functions are

$$\begin{aligned} U_p &= 2C\int_0^\infty e^{-a\omega} J_0(\omega\rho)\cos(\omega t)d\omega, \\ &= \frac{C}{[(a-it)^2+\rho^2]^{1/2}} + \frac{C}{[(a+it)^2+\rho^2]^{1/2}}, \end{aligned} \tag{6}$$

$$\begin{aligned} K_p = \frac{1}{2}C^2\{ & a^{-2} - \rho^2[(a-it)^2+\rho^2]^{-2} - \rho^2[(a+it)^2+\rho^2]^{-2} \\ & - a^{-2}(t^2+a^2-\rho^2)[t^4+2t^2(a^2-\rho^2)+(a^2+\rho^2)]^{-\frac{1}{2}}\}. \end{aligned} \tag{7}$$

3. Gravito-electromagnetism and invariants of the Weyl tensor

Now we introduce the well-known electric and magnetic Weyl tensors

$$X_{\alpha\beta} = C_{\mu\alpha\nu\beta}\tau^\mu\tau^\nu, \qquad Y_{\alpha\beta} = -\overset{*}{C}_{\mu\alpha\nu\beta}\tau^\mu\tau^\nu, \tag{8}$$

where C is the Weyl conformal curvature tensor, the star over indices denotes dual conjugation and τ is a unit time-like 4-covector. There are only four independent algebraic invariants of the Weyl tensor and they can be expressed through the fields X and Y

$$\begin{aligned} I_1 &= C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = 8(X_{\beta\delta}X^{\beta\delta} - Y_{\beta\delta}Y^{\beta\delta}), \\ I_2 &= \overset{*}{C}_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = -16X_{\beta\delta}Y^{\beta\delta}, \\ I_3 &= C_{\alpha\beta\gamma\delta}C^{\alpha\beta\grave{o}\eta}C_{\grave{o}\eta}{}^{\gamma\delta} = 16X^\alpha_\beta(X^\beta_\gamma X^\gamma_\alpha - 3Y^\beta_\gamma Y^\gamma_\alpha), \\ I_4 &= \overset{*}{C}_{\alpha\beta\gamma\delta}C^{\alpha\beta\grave{o}\eta}C_{\grave{o}\eta}{}^{\gamma\delta} = 16Y^\alpha_\beta(Y^\beta_\gamma Y^\gamma_\alpha - 3X^\beta_\gamma X^\gamma_\alpha). \end{aligned} \tag{9}$$

Since the quadratic constructions exclusively formed by X o Y are positive definite, it is clear that the sign of I_1 determines the electric type (plus) or magnetic (minus) of the field, whereas $I_1 = 0$ corresponds to the null type. The field is purely electric if also $I_2 = 0$, $I_4 = 0$, on the other hand the field is purely magnetic if also $I_2 = 0$, $I_3 = 0$. In this work we have softened the purity condition described by Mitskievich in [5], requiring only the vanishing of two invariants rather than three. We propose to call ultrapure fields those for whom a third invariant vanishes ($I_2 = 0$, $I_3 = 0$, $I_4 = 0$). When we conclude that the field is purely electric or purely magnetic, means that the alternative field can be transformed away by a choice of the reference frame.

The Einstein-Rosen solution has only two non-vanishing invariants:

$$\begin{aligned}I_1 = \frac{16}{\rho^2} e^{4(U-K)} (&U'^2 - 6\rho^3 U'U''\dot{U}^2 + 3\rho U'\dot{U}^2 + \rho U'U'' - 2\rho^3 U''U'^3 + 3\rho^2 U''\dot{U}^2 \\ &+ 3\rho^2 U''U'^2 - 9\rho^2 \dot{U}^2 U'^2 + 3\rho^4 U'^2 \dot{U}^4 - 3\rho^4 \dot{U}^2 U'^4 - 3\rho^3 U'\dot{U}^4 + 6\rho^3 \dot{U}^2 U'^3 \\ &+ 2\rho^3 \dot{U}'\dot{U}^3 + \rho^2 U''^2 + 2\rho^2 U'^4 + \rho^4 U'^6 - 3\rho^3 U'^5 + 3\rho^2 \dot{U}^4 - \rho^2 \dot{U}'^2 - \rho^4 \dot{U}^6 \\ &- 6\rho^2 \dot{U}U'\dot{U}' + 6\rho^3 \dot{U}\dot{U}'U'^2).\end{aligned} \tag{(10)}$$

$$\begin{aligned}I_3 = \frac{48}{\rho^2} e^{6(U-K)} [(&\rho U'^2 - U' - \rho \dot{U}^2)(3\rho^3 U'^2 \dot{U}^4 - \rho^3 \dot{U}^6 - 3\rho^2 U'\dot{U}^4 \\ &+ 2\rho \dot{U}^4 + 2\rho^2 \dot{U}'\dot{U}^3 - 3\rho^3 \dot{U}^2 U'^4 + 6\rho^2 \dot{U}^2 U'^3 - 6\rho^2 U'U''\dot{U}^2 - 7\rho \dot{U}^2 U'^2 \\ &+ 3\rho U''\dot{U}^2 + U'\dot{U}^2 + 6\rho^2 \dot{U}\dot{U}'U'^2 - 6\rho \dot{U}U'\dot{U}' + \rho^3 U'^6 - 2\rho^2 U''U'^3 \\ &- 3\rho^2 U'^5 + \rho U''^2 - \rho \dot{U}'^2 + \rho U'^4 + 3\rho U''U'^2 + U'U'' + 2U'^3)].\end{aligned} \tag{11}$$

The sign in them depends only on the derivatives of $U(t,\rho)$. We analyze the sign for a specific $U(t,\rho)$ more in detail below.

4. Comoving observers

Let us now introduce the monad field

$$\tau = \tau_{(0)}\theta^{(0)} + \tau_{(1)}\theta^{(1)}, \tag{12}$$

where $\theta^{(0)} = e^{(K-U)}dt$ and $\theta^{(1)} = e^{(K-U)}d\rho$ are elements of the tetrad basis in (1), coordinates will be numbered as $x^0 = t, x^1 = \rho, x^2 = \varphi, x^3 = z$. The co-vector τ can be used as the 4-velocity of the observer comoving with the wave when one of the electric or magnetic fields in this spacetime can be transformed away. The electric Weyl tensor has the components

$$\begin{aligned}&\mathrm{X}_{(0)(0)} = Q(t,\rho)[\tau^{(1)}]^2, \quad \mathrm{X}_{(1)(1)} = Q(t,\rho)[\tau^{(0)}]^2, \quad \mathrm{X}_{(0)(1)} = -Q(t,\rho)\tau^{(0)}\tau^{(1)}, \\ &\mathrm{X}_{(2)(2)} = R(t,\rho)[\tau^{(0)2} + \tau^{(1)2}] + S(t,\rho)\tau^{(0)}\tau^{(1)} + Q(t,\rho)[\tau^{(1)}]^2, \\ &\mathrm{X}_{(3)(3)} = -R(t,\rho)[\tau^{(0)2} + \tau^{(1)2}] - S(t,\rho)\tau^{(0)}\tau^{(1)} - Q(t,\rho)[\tau^{(0)}]^2,\end{aligned} \tag{13}$$

where

$$\begin{aligned}&Q(t,\rho) := \left(U'^2 - \rho^{-1}U' - \dot{U}^2\right)e^{2(U-K)}, \\ &R(t,\rho) := \left(\rho\ U'^3 - U'' + 3\rho\ \dot{U}^2 U' - 2U'^2 - \dot{U}^2\right)e^{2(U-K)}, \\ &S(t,\rho) := \left(2\rho\dot{U}^3 - 2\dot{U}' - 6\dot{U}U' + 6\rho\dot{U}U'^2\right)e^{2(U-K)}.\end{aligned}$$

The only non-vanishing component of the magnetic Weyl tensor

$$Y_{(2)(3)} = M(\rho,t)[(\tau^{(0)})^2 + (\tau^{(1)})^2] + N(\rho,t)\tau^{(0)}\tau^{(1)}, \tag{14}$$

where

$$\begin{aligned}&M(\rho,t) := \left(3\dot{U}U' - 3\rho\dot{U}U'^2 - \rho\dot{U}^3 + \dot{U}'\right)e^{2(U-K)}, \\ &N(\rho,t) := \left(3\dot{U}^2 + 3U'^2 + 2U'' + \rho^{-1}U' - 6\rho U'\dot{U}^2 - 2\rho U'^3\right)e^{2(U-K)}.\end{aligned}$$

It can be found the observer for the vanishing of Y but not for all components of X. Furthermore, in order to define a comoving observer, we have to introduce the super-Poynting vector

$$\mathrm{P}_\alpha = b_{\alpha\gamma} T^{\gamma\beta\lambda\mu} \tau_\beta \tau_\lambda \tau_\mu \equiv 2b_\alpha^{\text{ù}} E_{\text{ù}\lambda}^{\sigma\gamma} X^{\lambda\nu} Y_{\gamma\nu} \tau_\sigma, \tag{15}$$

where $T^{\gamma\beta\lambda\mu}$ is the tensor of Bel-Robinson, $b_{\alpha\gamma}$ projects onto the 3-subspace orthogonal to τ_μ and $E_{\text{ù}\lambda}^{\sigma\gamma}$ is the axial tensor of Levi-Civita. This vector represents the flux of super-energy of the gravitational field and we can define a comoving observer as that for whom the super-Poynting vector vanishes. Using the monad field (12), the super Poynting vector can be reduced for the Einstein-Rosen spacetime to

$$\mathrm{P}_{(\alpha)} = 2b_{(\alpha)(\sigma)}[X_{(2)(2)} - X_{(3)(3)}]Y^{(3)}_{(2)} E^{(\text{ù})(3)(\sigma)(2)} \tau_{\text{ù}}\,. \tag{16}$$

Then two comoving observers arise, the first one can be obtained of the vanishing of $Y^{(3)}_{(2)}$ and the second one can be obtained of the vanishing of the expression $\mathrm{X}_{(2)(2)}$-$\mathrm{X}_{(3)(3)}$. Using $\tau^{(0)} = \cosh(\psi)$ and $\tau^{(1)} = \sinh(\psi)$ as components of the monad field (12) with ψ being a function to be determined, the two comoving observers satisfy the relations

$$\frac{\tanh\psi_{II}}{1+\tanh^2\psi_{II}} = -\frac{M(\rho,t)}{N(\rho,t)},\ \frac{\tanh\psi_I}{1+\tanh^2\psi_I} = -\frac{2R(\rho,t)+Q(\rho,t)}{2S(\rho,t)}\,. \tag{17}$$

5. Propagation speed

Now we are in position to obtain the 3-velocity of propagation of the comoving observers with respect to the frame locally at rest $\theta^{(0)}$

$$\begin{aligned} v_{(\mu)} &= \frac{1}{\tau\cdot\tilde{\tau}}\tilde{b}_{(\mu)(\nu)}\tau^{(\nu)} = \frac{1}{\tau\cdot\tilde{\tau}}\left[g_{(\mu)(\nu)} - \tilde{\tau}_{(\mu)}\tilde{\tau}_{(\nu)}\right]\tau^{(\nu)}, \\ &= \frac{1}{\tau\cdot\tilde{\tau}}\left[\tau_{(\mu)} - (\tau\cdot\tilde{\tau})\tilde{\tau}_{(\mu)}\right] = -\tanh(\psi)\delta^1_\mu. \end{aligned}$$

And since $v = \tanh(\psi)$, we can write

$$\begin{aligned} v_I &= \tanh\left[\frac{1}{2}\operatorname{arctanh}\left|-\frac{2R(\rho,t)+Q(\rho,t)}{S(\rho,t)}\right|\right], \\ v_{II} &= \tanh\left[\frac{1}{2}\operatorname{arctanh}\left|-2\frac{M(t,\rho)}{N(t,\rho)}\right|\right]. \end{aligned} \tag{18}$$

The invariant I_1 of the Weyl tensor for the monochromatic wave takes positive values and since $I_2 = 0$ this field is of purely electric type. The 3-velocities with respect to the symmetry axis of both comoving observers for whom the super-Poynting vector vanishes are shown in Figure 1. The observer with velocity v_I propagates with the impure field of the wave whereas the observer with velocity v_{II} propagates with the purely electric part.

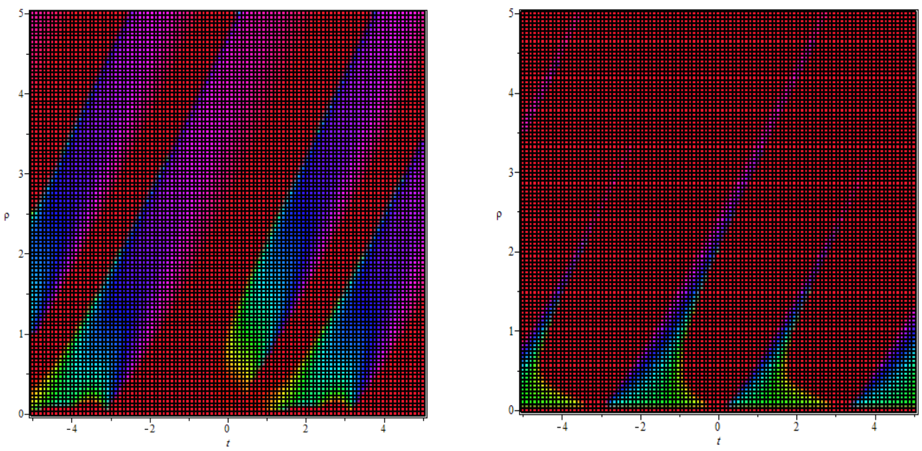

Figure 1. Velocity magnitudes v_I (left) and v_{II} (right) of the Einstein-Rosen monochromatic wave. The wave takes velocities near the speed of light on the magenta region and diminishes its speed near to zero as it approaches the orange region. Constants A and ω were set equal to 1.

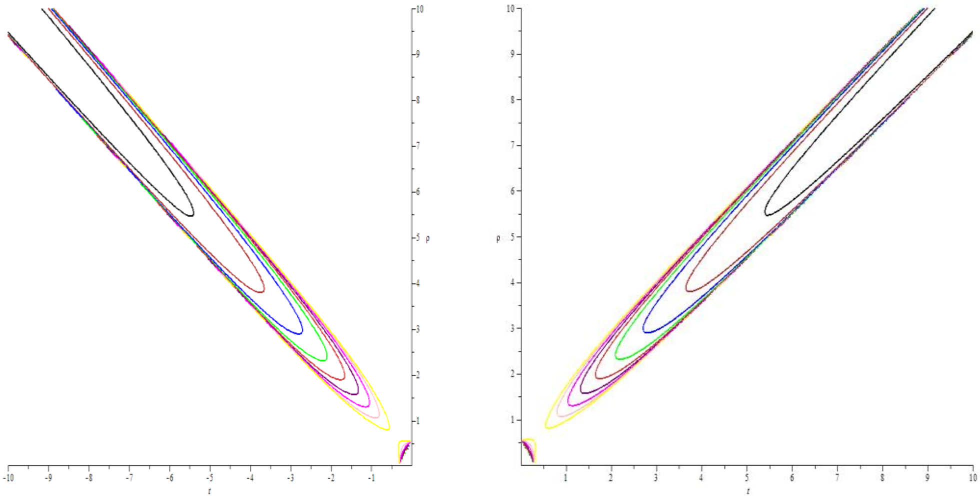

Figure 2. Curves of constant velocities of the Weber-Wheeler-Bonnor pulse for $v_I(\rho,t)$. Negative values are on the left, black represents $v_I=-0.8$, brown $v_I=-0.7$, blue $v_I=-0.6$, green $v_I=-0.5$, orange $v_I=-0.4$, purple $v_I=-0.3$, magenta $v_I=-0.2$, pink $v_I=-0.1$, and yellow $v_I=0$. Positive values are on the right, black represents $v_I=0.8$, brown $v_I=0.7$, blue $v_I=0.6$, green $v_I=0.5$, orange $v_I=0.4$, purple $v_I=0.3$, magenta $v_I=0.2$, pink $v_I=0.1$, and yellow $v_I=0$. Constants *a* and *C* were set equal to 1.

The same analysis is applied to the Weber-Wheeler-Bonnor pulse. The invariant I_1 of the Weyl tensor takes in this case positive, negative or zero values and since $I_2=0, I_4=0$, this field is of purely electric, purely magnetic or purely null type. The 3-velocities with respect to the symmetry axis of both comoving observers for whom the super-Poynting vector vanishes are shown in Figure 2 and 3. The observer with velocity v_I propagates with the impure field of the pulse whereas the observer with velocity v_{II} propagates with the purely electric part. The pulse slows down as it approaches the symmetry axis and then is reflected. Both observers are mutually complementary and fill all spacetime.

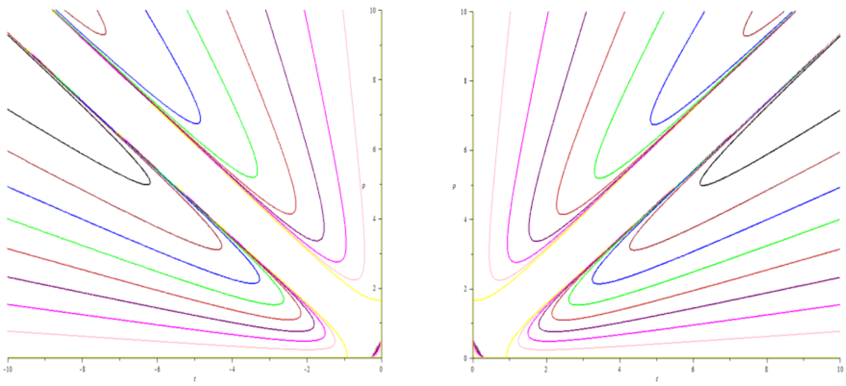

Figure 3. Curves of constant velocities of the Weber-Wheeler-Bonnor pulse for $v_{II}(\rho,t)$. Negative values are on the left, black represents $v_{II}=-0.8$, brown $v_{II}=-0.7$, blue $v_{II}=-0.6$, green $v_{II}=-0.5$, orange $v_{II}=-0.4$, purple $v_{II}=-0.3$, magenta $v_{II}=-0.2$, pink $v_{II}=-0.1$, and yellow $v_{II}=0$. Positive values are on the right, black represents $v_{II}=0.8$, brown $v_{II}=0.7$, blue $v_{II}=0.6$, green $v_{II}=0.5$, orange $v_{II}=0.4$, purple $v_{II}=0.3$, magenta $v_{II}=0.2$, pink $v_{II}=0.1$, and yellow $v_{II}=0$. Constants *a* and *C* were set equal to 1.

6. Concluding remarks

We have shown that it is possible to find the comoving frame for the Einstein-Rosen gravitational waves and therefore they propagate at sub-luminal speed in vacuum. There is a hole family of independent but complementary observers that fill all E-R spacetime for whom the super-Poynting vector vanishes, they propagate with different speeds, a part of them is comoving just with the gravito-electric field since the gravito-magnetic field vanishes completely in their frames of reference.

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