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Reconstructing f(R) modified gravity with dark energy parametrization

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Abstract. We demonstrate the reconstruction of $f(R)$ modified gravity theory with late-time accelerated cosmic expansion. A second-order differential equation for Lagrangian density is obtained from the field equation, and is solved as a function of the cosmic scale factor in two cases. First we begin with the case of a wCDM cosmological model, in which a dark-energy equation-of-state parameter w is constant, for simplicity. Next we extend the method to a case in which the parameter w is epoch-dependent and is expressed as the Chevallier-Polarski-Linder parametrization. Thus we can represent Lagrangian density of $f(R)$ modified gravity theory in terms of dark energy parameters.

1. Introduction

The discovery of accelerated expansion in the late-time universe by observations of distant type Ia supernovae [1, 2, 3] is manifestly one of the most remarkable advances in modern cosmology. Combining it with observational data of other astrophysical objects such as the cosmic microwave background, the universe acceleration is being established, but the theoretical solution to the cause of the acceleration is yet to be seen although theoretical possibilities have been explored with a large number of dark energy models [4, 5]. A '$f(R)$ modified gravity' dark energy model, in which some function of the scalar curvature R is added to the Einstein-Hilbert action as a correction, is one of such theoretically possible models [6]. It is of great interest for theorists to study this model because it is, in a sense, most straightforward extension of Einstein's general relativity, and also will provide a clarification of the link between the universe acceleration and gravitational theories. (We should bear in mind the role of Starobinsky's R^2-model in inflationary cosmology [7].) There is, however, no guiding principle for determining the form of $f(R)$; only severe constraints have been obtained from local-scale experiments (e.g. solar system tests) [8] and cosmological observations [9, 10, 11]. Therefore, we can only *design* the form of $f(R)$ so that it satisfies those observational constraints.

In this article, we demonstrate the reconstruction of $f(R)$ modified gravity theory for a given accelerated cosmic expansion as an illustration of designing $f(R)$. As models of the cosmic expansion, we employ a wCDM cosmological model, in which a dark-energy equation-of-state parameter w is constant, and the Chevallier-Polarski-Linder (CPL) parametrization [12, 13], which has been frequently used in dark energy phenomenology. This work extends the previous works that treat the reconstruction of $f(R)$ gravity with mainly a ΛCDM model [14, 15, 16].

2. Basic equations in $f(R)$ modified gravity

We consider $f(R)$ modified gravity theory with the action

$$\mathcal{S} = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\ \mathrm{d}^4x\ + \mathcal{S}_\mathrm{m}\,, \tag{1}$$

where G is Newton's gravitational constant, R is the four-dimensional scalar curvature, g is the determinant of the space-time metric $g_{\mu\nu}$, and $\mathcal{S}_\mathrm{m}$ denotes the action of matter field described as a pressureless fluid. The action principle yields the field equations

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = 8\pi G T^{(\mathrm{m})}_{\mu\nu}\,, \tag{2}$$

where $F(R) := \partial_R f(R)$, ∇ represents the four-dimensional covariant derivative, $\Box := g^{\mu\nu}\nabla_\mu\nabla_\nu$, and $T^{(\mathrm{m})}_{\mu\nu}$ is the energy-momentum tensor of pressureless matter. Hereafter we write the function $f(R)$ as $R - \xi(R)$, where $\xi(R)$ plays the role of an effective dark energy component, and then $F(R) = 1 - \partial_R\xi(R)$.

In a homogeneous and isotropic universe model with the scale factor $a(t)$, a spatial constant curvature K, and the energy density $\rho_\mathrm{M}(t)$ of pressureless matter, the $(0,0)$-component of the field equations (2) reads

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho_\mathrm{M} + \frac{1}{6}\xi(R) + H\partial_t\left(\partial_R\xi(R)\right) - \partial_R\xi(R)\left(\partial_t H + H^2\right)\,, \tag{3}$$

where $H := \partial_t a/a$ is the Hubble parameter, and $R = 6\partial_t H + 12H^2 + 6K/a^2$. Note that this equation is reduced to the Friedmann equation in the standard FLRW universe model if we set $\xi(R) = 2\Lambda$ with a positive cosmological constant Λ. Replacing the time variable t with a, and the variable R with a, and rearranging Eq. (3), we obtain

$$\partial_a^2\xi - \left(\frac{1}{a} + \frac{\partial_a H}{H} + \frac{\partial_a^2 R}{\partial_a R}\right)\partial_a\xi + \frac{\partial_a R}{6aH^2}\xi = \frac{\partial_a R}{aH^2}\left(H^2 + \frac{K}{a^2} - \frac{8\pi G}{3}\rho_\mathrm{M}\right)\,. \tag{4}$$

3. Reconstruction of $f(R)$ gravity with dark energy parametrization

We assume that the cosmic expansion is that of the universe with a spatial constant curvature k, matter density $\rho_\mathrm{m}(t)$, and dark energy density $\rho_\mathrm{d}(t)$, obeying Einstein's general relativity. Then the Friedmann equation is written as

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\rho_\mathrm{m} + \rho_\mathrm{d}\right)\,. \tag{5}$$

Introducing the dark-energy equation-of-state parameter $w := P_\mathrm{d}/\rho_\mathrm{d}$ with the dark energy pressure P_d, the integration of the continuity equation gives the dark energy density as

$$\rho_\mathrm{d}(a) = \rho_\mathrm{d}(a_0)\,\mathcal{E}(a)\,; \qquad \mathcal{E}(a) := \exp\left[-3\int_{a_0}^{a}\frac{1+w(\tilde{a})}{\tilde{a}}\,\mathrm{d}\tilde{a}\right]\,, \tag{6}$$

where the subscript '0' denotes the present value. Then the Friedmann equation (5) is rewritten as $H^2 = H_0^2\left(\Omega_{\mathrm{m}0}\,a^{-3} + \Omega_{\mathrm{d}0}\mathcal{E} + \Omega_{k0}\,a^{-2}\right)$, where $\Omega_{\mathrm{m}0} := 8\pi G\rho_\mathrm{m}(a_0)/(3H_0^2)$, $\Omega_{\mathrm{d}0} := 8\pi G\rho_\mathrm{d}(a_0)/(3H_0^2)$, $\Omega_{k0} := -k/H_0^2$ and $a_0 := 1$, and the scalar curvature as

$$R = 3H_0^2\left[\Omega_{\mathrm{m}0}\,a^{-3} + (1-3w)\Omega_{\mathrm{d}0}\mathcal{E} + 2\left(\Omega_{k0} - \Omega_{K0}\right)a^{-2}\right]\,, \tag{7}$$

where $\Omega_{K0} := -K/H_0^2$. Rewriting the right-hand side of Eq. (4), we arrive at the following key equation for the reconstruction:

$$\partial_a^2\xi - \left(\frac{1}{a} + \frac{\partial_a H}{H} + \frac{\partial_a^2 R}{\partial_a R}\right)\partial_a\xi + \frac{\partial_a R}{6aH^2}\xi = \frac{\partial_a R}{aH^2}H_0^2\Big[\left(\Omega_{k0} - \Omega_{K0}\right)a^{-2} + \left(\Omega_{\rm m0} - \Omega_{\rm M0}\right)a^{-3} + \Omega_{\rm d0}\mathcal{E}\Big]\,, \tag{8}$$

where $\Omega_{\rm M0} := 8\pi G\rho_{\rm M}(a_0)/(3H_0^2)$. Hereafter, we assume that $\Omega_{m0} = \Omega_{M0}$ and $\Omega_{k0} = \Omega_{K0} = 0$ for simplicity.

3.1. Case of the wCDM model

First we demonstrate the reconstruction in a wCDM model, in which the parameter w is constant, and then $\mathcal{E}(a) = a^{-3(1+w)}$. The key equation (8) is reduced to

$$\partial_a^2\xi - \frac{1}{a}C_1(a)\partial_a\xi - \frac{3}{2a^2}C_2(a)\xi = -\frac{9}{a^2}C_2(a)H_0^2\Omega_{\rm d0}\,a^{-3(1+w)}\,, \tag{9}$$

where

$$C_1(a) := -\frac{9}{2} - \frac{3w}{2}\frac{\Omega_{\rm d0}}{\Omega_{\rm m0}}a^{-3w}\left[\left(1 + \frac{\Omega_{\rm d0}}{\Omega_{\rm m0}}a^{-3w}\right)^{-1} + 2(1-3w)(1+w) \times\left(1 + (1-3w)(1+w)\frac{\Omega_{\rm d0}}{\Omega_{\rm m0}}a^{-3w}\right)^{-1}\right]\,, \tag{10}$$

$$C_2(a) := 1 - (2w + 3w^2)\frac{\Omega_{\rm d0}}{\Omega_{\rm m0}}a^{-3w}\left(1 + \frac{\Omega_{\rm d0}}{\Omega_{\rm m0}}a^{-3w}\right)^{-1}\,. \tag{11}$$

To find homogeneous solutions of Eq. (9) in an approximate form, we make an ansatz $\xi(a) \propto a^n$, and then find that $n = \left(1 + C_1 \pm \sqrt{(1+C_1)^2 + 6C_2}\right)/2 =: n_\pm$, which asymptotically becomes $n_\pm \approx \left(-7 \pm \sqrt{73}\right)/4$ in matter-dominated era ($a^{-3w} \ll 1$), and $n_\pm \approx \left(-7 - 9w \pm \sqrt{73 + 78w + 9w^2}\right)/4$ in dark-energy-dominated era ($a^{-3w} \gg 1$). Since $R \approx 3H_0^2\Omega_{\rm m0}\,a^{-3}$ and $R \approx 3(1-3w)H_0^2\Omega_{\rm d0}\,a^{-3(1+w)}$ in these respective eras, the homogeneous solutions lead to $\xi(R) \propto R^{(7+\sqrt{73})/12} \approx R^{1.3}$ in matter-dominated era, and $\xi(R) \propto R^{(7+9w-\sqrt{73+78w+9w^2})/(12+12w)} \approx R^{-3.6}$ (for $w \approx -0.9$) in dark-energy-dominated era. These have been found to be observationally inconsistent [9, 10], and thus we take only the particular solution of Eq. (9) into account, omitting the homogeneous solutions. Using the method of a Green function, we obtain

$$\xi \approx \frac{6H_0^2\Omega_{\rm d0}\,a^{-3(1+w)}}{2 - 5w - 6w^2} \approx \frac{6H_0^2\Omega_{\rm d0}}{2 - 5w - 6w^2}\left(\frac{R}{3H_0^2\Omega_{\rm m0}}\right)^{1+w} \quad \text{in matter-dominated era,} \tag{12}$$

$$\xi \approx 3(1-3w)H_0^2\Omega_{\rm d0}\,a^{-3(1+w)} \approx R \qquad \text{in dark-energy-dominated era.} \tag{13}$$

3.2. Case of the CPL parametrization

Next we proceed to a more general case, in which the parameter w is expressed as $w = w_0 + w_1(1-a)$ with constants w_0 and w_1, and then $\mathcal{E}(a) = a^{-3(1+w_0+w_1)}\exp[-3w_1(1-a)]$. This phenomenological model is known as the Chevallier-Polarski-Linder (CPL) parametrization. Here we focus on finding the form of $\xi(R)$ only in matter-dominated era because the universe is considered to have been matter dominant during most of the epoch after photon decoupling, except for the most recent epoch. Then the key equation (8) becomes

$$\partial_a^2\xi + \frac{9}{2a}\partial_a\xi - \frac{3}{2a^2}\xi = -\frac{9H_0^2\Omega_{\mathrm{d}0}}{a^2}a^{-3(1+w_0+w_1)}\exp[-3w_1(1-a)]\,, \tag{14}$$

whose particular solution is

$$\xi = \frac{-18H_0^2\Omega_{\mathrm{d}0}\,\mathrm{e}^{-3w_1}}{\sqrt{73}}\left[a^{n_+}\int^a \tilde{a}^{-(n_++4+3w_0+3w_1)}\mathrm{e}^{3w_1\tilde{a}}\,\mathrm{d}\tilde{a} - a^{n_-}\int^a \tilde{a}^{-(n_-+4+3w_0+3w_1)}\mathrm{e}^{3w_1\tilde{a}}\,\mathrm{d}\tilde{a}\right], \tag{15}$$

where $n_\pm = (-7 \pm \sqrt{73})/4$. Utilizing the Taylor expansion for $\mathrm{e}^{3w_1 a}$, we obtain

$$\xi = \sum_\ell \frac{-18H_0^2\Omega_{\mathrm{d}0}\,\mathrm{e}^{-3w_1}a^{-3(1+w_0+w_1)}}{2\,[\ell-3(1+w_0+w_1)]^2+7\,[\ell-3(1+w_0+w_1)]-3}\frac{(3w_1 a)^\ell}{\ell!} \tag{16}$$

$$\approx \sum_\ell \frac{-18H_0^2\Omega_{\mathrm{d}0}\,\mathrm{e}^{-3w_1}(3w_1)^\ell}{\ell!\left\{2\,[\ell-3(1+w_0+w_1)]^2+7\,[\ell-3(1+w_0+w_1)]-3\right\}}\left(\frac{R}{3H_0^2\Omega_{\mathrm{m}0}}\right)^{1+w_0+w_1-\ell/3}. \tag{17}$$

4. Summary and outlook

In this article, the reconstruction of Lagrangian density of $f(R)$ gravity has been demonstrated for a given cosmic expansion in the context of dark energy cosmology. We have adopted a wCDM model, in which dark-energy equation-of-state parameter w is constant, and the CPL parametrization, as phenomenological models of accelerated cosmic expansion. In these cases, Lagrangian density of $f(R)$ gravity has been expressed in terms of dark energy parameters. We hope that the results obtained would be useful in clarifying the relation between dark energy parameters and local gravity constraints, and also in evaluating the effect of gravity modification on cosmological perturbations in the framework of $f(R)$ gravity theory.

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