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The origin of discrete symmetries in F-theory models

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Abstract. While non-abelian groups are undoubtedly the cornerstone of Grand Unified Theories (GUTs), phenomenology shows that the role of abelian and discrete symmetries is equally important in model building. The latter are the appropriate tool to suppress undesired proton decay operators and various flavour violating interactions, to generate a hierarchical fermion mass spectrum, etc. In F-theory, GUT symmetries are linked to the singularities of the elliptically fibred K3 manifolds; they are of ADE type and have been extensively discussed in recent literature. In this context, abelian and discrete symmetries usually arise either as a subgroup of the non-abelian symmetry or from a non-trivial Mordell-Weil group associated to rational sections of the elliptic fibration. In this note we give a short overview of the current status and focus in models with rank-one Mordell-Weil group.

1. Introduction

Discrete symmetries play a vital role in model building [1, 2, 3, 4]. Over the past few decades they have been widely used to restrict the superpotential and suppress the exotic interactions of numerous proposed effective theories. In the Standard Model, as well as in old GUTs, abelian factors and discrete symmetries of Z_N type were imposed to forbid dangerous Lepton and Baryon number violating operators. In more recent scenarios, non-abelian discrete groups were introduced to interpret the mixing properties of the neutrino sector [5, 6, 7]. In the field theory context, these symmetries were postulated purely on phenomenological grounds. However, it is not clear whether such global (including discrete) symmetries can exist [8]. In this respect, it would be interesting to investigate whether such symmetries can be justified in the context of string theory. Recently, considerable work in this direction has been devoted [9]–[15]. A fascinating possibility in particular arises in F-theory constructions where symmetries are tightly connected to the elliptically fibred internal space. This compact space is a four-dimensional complex manifold (a fourfold) while the gauge symmetries are linked to its singularities. Hence, we may consider that all symmetries, including the discrete part, are associated to the geometric properties of the fourfold. In the present talk I will discuss the origin of discrete symmetries in F-models. I will start with a short description of the basic features of F-model building focusing in particular to the properties related to the elliptic curves.

2. Elliptic Curves

F-theory [16] is an exciting reformulation of String Theory in a 12-dimensional space which consists of the 4 space-time dimensions and an 8-dimensional internal elliptically fibred compact space. Because of its relation to the theory of elliptic curves and in particular their complex representations, effective F-theory models are endowed with many interesting properties. In the



present talk I will mainly focus on some features of the abelian and discrete symmetries that emerge from the rational sections of the elliptic curves.

Many of the properties that will be discussed are related to rational points on curves. A point is said to be rational if its coordinates are rational, while a rational curve is defined by an equation with rational coefficients. It is trivial to find the rational points on lines and conics. Consider for example the equation of the unit circle $x^2 + y^2 = 1$. Choosing a rational point on it, -let it be $(-1, 0)$ - we can draw a line which intersects the circle at (x, y) . We can map this pair on the line identified with the vertical axis y by the transformation

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2},$$

This way, all rational points (x, y) on the curve can be determined in terms of the rational

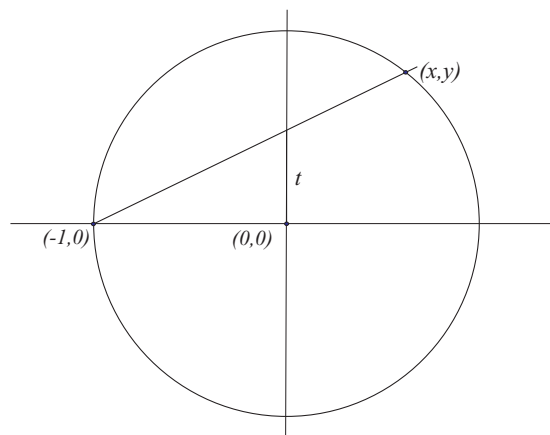


Figure 1. Rational points on conics.

values of the parameter t . Because of this correspondence between the rational points, we say that the curve (in this case the conic) is **birationally equivalent** to the line. A rational curve is also called a curve of genus zero. Every genus zero curve is birationally equivalent either to a conic or to a line.

We proceed now to the elliptic curves which are described by a cubic equation whose most general form can be written as

$$\mathcal{C} : \sum_{n=0}^3 \sum_{m=0}^n a_{m,n} x^n y^{n-m} = 0 \quad (1)$$

The identification of the rational points on a given elliptic curve \mathcal{C} of general type is much more complicated compared to the conic. We know however, that the rational points of \mathcal{C} exhibit a group structure. According to Mordell's theorem,

If a non-singular elliptic curve \mathcal{C} has a rational point then the group of rational points can be finitely generated.

In other words, there is a finite number of elements generating the whole group. The group structure is depicted here in figure 2 which can be defined as follows: Let P, Q two rational points on \mathcal{C} . Drawing the line joining these two points, we can find another one at the third intersection of the line with the curve \mathcal{C} . I designate this point with $P * Q$. Suppose now we are given a rational point O on \mathcal{C} that we can identify this to be the zero element of the group. The

line from O to $P * Q$ intersects \mathcal{C} on another point which, as can be proved ¹, under the group law is the point $P + Q$. To find the opposite element with respect to the addition law of the group, we draw a tangent to the zeroth element O which intersects \mathcal{C} at some point called here S . One can prove that the opposite to P is identified with the third intersection of the PS -line and curve \mathcal{C} , so that $P + (-P) = O$.

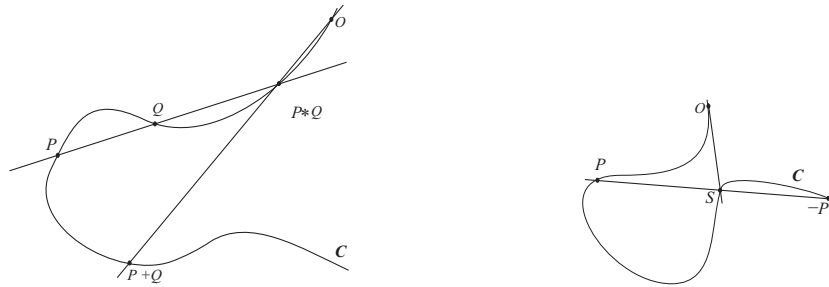


Figure 2. The group structure of rational points on elliptic curves. The law of addition.

The general form (1) of the elliptic equation is rather too complicated. Fortunately it can be shown that any cubic equation with a rational point on it can be brought to the Weierstraß form

$$y^2 = x^3 + fx + g \quad (2)$$

We can readily check that the Weierstraß form is symmetric with respect to the x -axis. Moreover, the zeroth element of the group can be taken to infinity while the sum of two points is just the reflection (w.r.t. x -axis) of the third intersection point of the line PQ with \mathcal{C} . There are two important quantities characterising the elliptic curves. These are:

- The discriminant

$$\Delta = 4f^3 + 27g^2 \quad (3)$$

which classifies the singularities on the matter curve. In particular, when $\Delta \neq 0$ curves are non-singular and may have one or three real roots. When $\Delta = 0$ curves are singular. Singularities are of nodal or cuspidal type.

- The j -invariant (modular invariant) function

$$j(\tau) = \frac{4(24f)^3}{4f^3 + 27g^2} = \frac{4(24f)^3}{\Delta} \quad (4)$$

which takes the same value for equivalent elliptic curves characterised by the $SL(2, Z)$ transformations $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$.

What happens when we consider complex coefficients (functions) $f, g \in \mathbf{C}$? In this case, it can be proved that a complex elliptic curve \mathcal{C} is a genus 1 closed surface with a marked point on it corresponding to its neutral element (point to infinity). Defining a modulus τ as usually, the torus, hence \mathcal{C} , is equivalent to the lattice $(1, \tau)$. With respect to the previous analysis of elliptic curves, we distinguish two cases: The complex analogue of a real elliptic curve with non-singular points is a torus without singularities. On the contrary, if the real elliptic curve has singular points then its complex equivalent is a torus with a pinched radius.

¹ See for example standard textbooks such as [17, 18, 19].

3. F-theory and Elliptic Fibration

In this section, a few basic features of F-theory [16] are described which are useful to the subsequent analysis. The shortest (however incomplete) definition of F-theory is that it is the geometrisation of the type II-B superstring. The type IIB string is distinguished by its closed string spectrum (which differs, say, from II-A case). It is obtained by combining L- and R-moving open strings with the truncated two types of boundary conditions, namely the Neveu-Schwarz (antiperiodic) and Ramond (periodic) boundary conditions.

In the bosonic spectrum there are two scalars, the dilaton field ϕ and a zero-form potential (axion) C_0 . One then can define a modulus, the axion-dilaton complex structure

$$\tau = C_0 + ie^{-\phi} \quad (5)$$

and write down an $SL(2, Z)$ invariant action of the ten-dimensional theory which leads to the correct equations of motion. The terms of the action seem as if they are obtained from a twelve dimensional theory compactified along the two radii of the torus (for a review see [20]). We can think of τ as the modulus of a torus attached to each point of the internal manifold of three complex dimensions (threefold), as depicted in figure 3. We end up with a fibred fourfold. Recalling the analysis of the first section, one is tempted to consider the interesting possibility

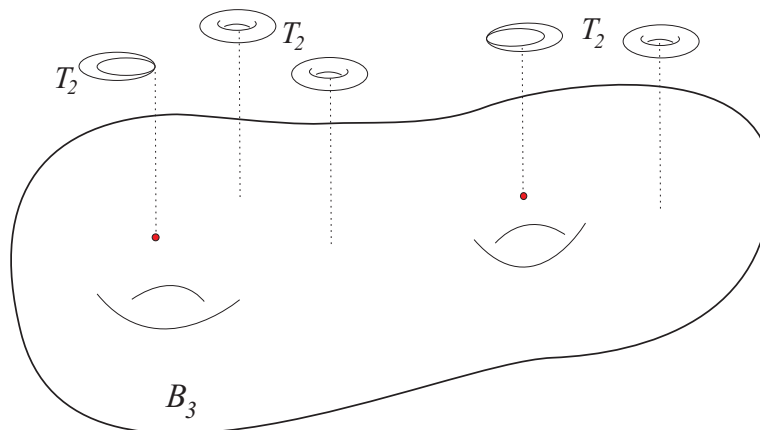


Figure 3. Elliptic fibration. At each point of the threefold B_3 a torus $\tau = C_0 + ie^{-\phi}$ is assigned.

of describing this fibration by the Weierstraß model given in equation (2). In particular, we write it in the form

$$y^2 = x^3 + fxz^4 + gz^6 \quad (6)$$

where x, y, z are homogeneous complex coordinates. As explained previously, for f, g complex, equation (2) describes a torus whose modulus τ is now identified with that of (5). On the other hand, in order to satisfy the Calabi-Yau (CY) conditions, we also require the two functions $f = f(w), g = g(w)$ to be 8^{th} and 12^{th} degree polynomials of the complex variable w . As we move from point to point in the internal manifold, the modulus τ varies. In particular, on moving along non-trivial closed cycles, τ undergoes non-trivial $SL(2, Z)$ transformations². In figure 3 for any generic point we draw a normal torus, while pinched torii are drawn at points of singularities; the latter appear when two D7-branes intersect at a ‘point’ of the manifold. These correspond to singularities of elliptic surfaces and were classified in terms of the vanishing orders of the discriminant and the polynomials $f(w), g(w)$ several decades ago by Kodaira [22]. For

² The $SL(2, Z)$ modular invariant function is given by $j(\tau) = e^{-2\pi i\tau} + 744 + \mathcal{O}(e^{-2\pi i\tau})$. Combined with (4) one can elaborate [21] a relation approximated with $\tau(w) \sim \frac{1}{2\pi i} \ln(w - w_i)$ in the vicinity of the zeros of $\Delta(w_i) = 0$.

Type	Group	a_1	a_2	a_3	a_4	a_6	Δ
I_0	—	0	0	0	0	0	0
I_1	—	0	0	1	1	1	1
I_2	$SU(2)$	0	0	1	1	2	2
I_{2m}^{ns}	$Sp(m)$	0	0	m	m	$2m$	$2m$
I_{2m}^s	$SU(2m)$	0	1	m	m	$2m$	$2m$
I_{2m+1}^s	$SU(2m+1)$	0	1	m	$m+1$	$2m+1$	$2m+1$
I_1^{*s}	$SO(10)$	1	1	2	3	5	7
I_{2m-3}^{*ns}	$SO(4m+1)$	1	1	m	$m+1$	$2m$	$2m+3$
I_{2m-3}^{*s}	$SO(4m+2)$	1	1	m	$m+1$	$2m+1$	$2m+3$
I_{2m-2}^{*ns}	$SO(4m+3)$	1	1	$m+1$	$m+1$	$2m+1$	$2m+4$
I_{2m-2}^{*n}	$SO(4m+4)^*$	1	1	$m+1$	$m+1$	$2m+1$	$2m+4$
IV^{*s}	\mathcal{E}_6	1	2	2	3	5	8
III^{*s}	\mathcal{E}_7	1	2	3	3	5	9
II^s	\mathcal{E}_8	1	2	3	4	5	10

Table 1. Selected cases of Tate’s algorithm. The first column declares the type of the singular fiber according to Kodaira, i.e. nodal (I_1), cuspidal (II) etc. The superscripts s, ns stand for *split* and *non-split*. (The complete results can be found in [34, 23, 24].) The other columns show the order of vanishing of the coefficients $a_i \sim z^{n_i}$, the discriminant Δ and the corresponding gauge group.

minimal elliptic surfaces eight types of singular fiber were identified, (nodal, cuspidal or otherwise reducible). The singularities are related to simply-laced Dynkin diagrams of ADE type. These extremely interesting results can be found in several recent papers and reviews [22, 23, 24], thus, they will not be presented in this short note. Instead, we will shortly give an analogous algorithm in another representation which will be useful in our subsequent analysis.

The nature of the singularities of the internal space motivated the idea that they can be identified with the gauge symmetries of the effective field theory model. If this is true, then one can attribute all the properties of the internal manifold to the massless spectrum and the effective potential describing their interactions. This scenario has many advantages, including calculability of Yukawa couplings [25]–[33] of the effective theory from a handful of geometric characteristics of the internal space.

A convenient description which emphasizes the local properties of these singularities is given in terms of Tate’s algorithm [34]. In this context, the equation describing the elliptically fibred space takes the form

$$y^2 + \alpha_1 xyz + \alpha_3 yz^3 = x^3 + \alpha_2 x^2 z^2 + \alpha_4 xz^4 + \alpha_6 z^6 \quad (7)$$

The variables $[x, y, z]$ have weights $[2 : 3 : 1]$ correspondingly, defining a hypersurface in the $\mathbf{P}_{(2,3,1)}$ weighted projective space.

In analogy with Kodaira’s classification of singularities, here also the gauge group is determined in terms of the vanishing orders of the polynomials α_k and the discriminant Δ . The results are summarised in Table 1. We note that the Weierstraß equation can be obtained from Tate’s form by recovering the functions f, g from the coefficients α_k . To this end, it is

convenient to define the following quantities

$$\beta_2 = \alpha_1^2 + 4\alpha_2; \beta_4 = \alpha_1\alpha_3 + 2\alpha_4; \beta_6 = \alpha_3^2 + 4\alpha_6; \beta_8 = \frac{1}{4}(\beta_2\beta_6 - \beta_4^2). \quad (8)$$

Then, it can be readily checked that the functions f, g and the discriminant Δ are

$$f = \frac{1}{48}(24\beta_4 - \beta_2^2) \quad (9)$$

$$g = \frac{1}{864}(\beta_2^3 - 36\beta_4\beta_2 + 216\beta_6) \quad (10)$$

$$\Delta = -8\beta_4^3 + 9\beta_2\beta_6\beta_4 - 27\beta_6^2 - \beta_2^2\beta_8 \quad (11)$$

After these preliminary notes, in the next section we proceed to the description of the basic tools for local model building.

3.1. GUT models with discrete symmetries

The attractive scenario of linking gauge symmetries to the singularities of the internal geometry leads to far reaching implications. An interesting advantage of F-theory constructions based on the elliptic fibration, is the appearance of the exceptional symmetry \mathcal{E}_8 where the gauge group of the effective theory is embedded [35, 36, 37, 38, 39]³. However, phenomenological investigations have shown that additional symmetries (discrete or continuous) are required to render the theory viable. Interestingly, a useful class of such symmetries originates from the commutant of this GUT with respect to the exceptional gauge symmetry \mathcal{E}_8 .

To show how these symmetries appear we describe the \mathcal{E}_6 and $SU(5)$ gauge groups in brief. In the local picture, Tate's coefficients have a general expansion of the form

$$\alpha_k = \alpha_{k0} + \alpha_{k1}w + \alpha_{k2}w^2 + \dots \quad (12)$$

If a certain coefficient α_k has vanishing order n , it is convenient to write

$$\alpha_k = \alpha_{k,n}w^n, \text{ with } \alpha_{k,n} = \alpha_{kn} + \alpha_{k(n+1)}w + \dots \quad (13)$$

Hence, for an \mathcal{E}_6 type of singularity, the coefficients take the form

$$\alpha_1 = \alpha_{1,1}w, \alpha_2 = \alpha_{2,2}w^2, \alpha_3 = \alpha_{3,2}w^2, \alpha_4 = \alpha_{4,3}w^3, \alpha_6 = \alpha_{6,5}w^5$$

With this choice, the discriminant is factorised as follows

$$\Delta = \Delta_0 w^8 \quad (14)$$

with

$$\Delta_0 = -27\alpha_{3,2}^4 + A(\alpha_{kj})w + \mathcal{O}(w^2) \quad (15)$$

where

$$A(\alpha_{kj}) = (\alpha_{1,1}\alpha_{3,2} + 2\alpha_{4,3}) \left((\alpha_{1,1}^2 + 36\alpha_{2,2}) \alpha_{3,2}^2 - 32\alpha_{4,3}(\alpha_{1,1}\alpha_{3,2} + \alpha_{4,3}) \right) - 216\alpha_{3,2}^2\alpha_{6,5}$$

Indeed, Δ has a vanishing order of 8^{th} degree, in accordance with Table 1. From (14) we observe that the discriminant locus consists of two divisors, $D_{\mathcal{E}_6}$ (at $w = 0$ of multiplicity eight) and $D_{\mathcal{I}}$

³ For reviews, see [40, 41, 42, 43]

(at $\Delta_0 = 0$ of multiplicity one). There are eight $D7$ branes wrapping the divisor $D_{\mathcal{E}_6}$ and one $D7$ brane wrapping $\Delta_{\mathcal{I}}$ which is assumed to be irreducible.

The representations of the effective theory model, reside at the intersections of the $D_{\mathcal{E}_6}$ divisor with $D7$ branes spanning different dimensions of the internal space. These intersections (often called matter curves) are in fact Riemann surfaces along which symmetry is enhanced. In the elliptic fibration the highest allowed singularity is \mathcal{E}_8 . Then, a convenient way to see the effective model is through the decomposition

$$\mathcal{E}_8 \rightarrow \mathcal{E}_6 \times SU(3)$$

where \mathcal{E}_6 is the desired GUT, while the enhancements along the matter curves include factors embedded in $SU(3)$. We can also think of $SU(3)$ broken by fluxes (or some other mechanism) to a subgroup of it. The possibilities are either the continuous symmetries $SU(2), U(1)^2$, or a discrete group such as the S_3 (permutation of three objects), Z_3 or Z_2 . Hence all \mathcal{E}_6 representations transform non-trivially under the latter. Viable cases have a final symmetry such as [38, 47, 48, 49]:

$$\mathcal{E}_6 \times U(1)^2, \mathcal{E}_6 \times S_3, \mathcal{E}_6 \times Z_2$$

As a second example we consider that the GUT gauge symmetry is associated to a divisor characterised by an $SU(5)$ singularity⁴, while the commutant is also $SU(5)$ -usually denoted with $SU(5)_{\perp}$. It's a simple exercise to repeat the above analysis for the $SU(5)$ case too. Instead, let us focus on another issue. A phenomenologically friendly description of these symmetries is based on the idea of the spectral cover. In this case the implications of $SU(5)_{\perp}$ are described by a spectral cover denoted by \mathcal{C}_5 and represented by the five degree polynomial of an affine coordinate s ,

$$\mathcal{C}_5 : \sum_{k=0}^5 b_k s^{5-k} = b_0 s^5 + b_1 s^4 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 = 0 \quad (16)$$

The coefficients b_k fulfill the conditions for the $SU(5)$ singularity ($b_1 = 0$ for $SU(n)$)

$$b_0 = \alpha_{6,5}, b_2 = \alpha_{4,3}, b_3 = \alpha_{3,2}, b_4 = \alpha_{2,1}, b_5 = \alpha_{1,0}$$

Equation (16) includes the basic information regarding geometric properties as well as additional symmetries of the $SU(5)$ F-GUT. Depending on the specific topological structure of the internal space, the spectral cover \mathcal{C}_5 may factorise in various ways. A few interesting cases are

$$\mathcal{C}_4 \times \mathcal{C}_1, \mathcal{C}_3 \times \mathcal{C}_2, \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_1$$

implying analogous factorisations of the polynomial (16). For the $SU(5)$ GUT, there is a rich variety of possible accompanying discrete symmetries, including [44, 50, 14]

$$SU(5) \times A_4 \times U(1), SU(5) \times Z_3 \times Z_2, SU(5) \times Z_2 \times Z_2 \times U(1)$$

4. Mordell-Weil $U(1)$'s and discrete symmetries

In the previous section we presented the classification of the non-abelian singularities of the elliptic fiber, subject to restrictions arising from Kodaira classification. Adopting the interpretation that these correspond to non-abelian gauge symmetries, we were able to determine the GUT gauge group of the potential F-theory models. There is considerable activity [52]-[73] on

⁴ F- $SU(5)$ models have been extensively discussed in the literature [36, 37, 38, 39, 44, 45, 46, 47, 48]

the role of the abelian sector related to the rational sections of the elliptic curves. Subsequently, we focus in some related issues.

In the introductory section we have seen that there is a class of abelian symmetries, associated to rational sections of the elliptic curves. Since the internal space is elliptically fibred, these $U(1)$'s may manifest themselves in potential low energy effective models. From analyses of phenomenological models, we know that such symmetries are extremely useful in order to prevent unwanted terms in the lagrangian. It seems that such abelian symmetries are indispensable when constructing an F-theory effective model, however, a Kodaira-type classification is lacking up to now. From the Mordell-Weil theorem we only know that these are related to the rational sections defined on elliptic curves but the rank of this group is not known. The Mordell-Weil group can be written as

$$\mathcal{E}(\mathcal{K}) = \underbrace{Z \oplus Z \oplus \cdots \oplus Z}_n \oplus \mathcal{G} \equiv Z^n \oplus \mathcal{G}$$

Here n is the rank of the Abelian group, while \mathcal{G} is the torsion subgroup and \mathcal{K} is the number field. According to a theorem by Mazur [53] (see also [54]), the possible torsion subgroups are either \mathbf{Z}_k , $k = 1, 2, \dots, 10, 12$ or the direct sum $\mathbf{Z}_2 \oplus \mathbf{Z}_{2k}$ with $k = 1, 2, 3, 4$.

$$\mathcal{G} = \begin{cases} \mathbf{Z}_n & n = 1, 2, \dots, 10, 12 \\ \mathbf{Z}_{2k} \oplus \mathbf{Z}_2 & k = 1, 2, 3, 4 \end{cases} \quad (17)$$

A specific choice of the coefficients in an elliptic curve equation eventually will fix the symmetries of the effective GUT model. For a simple demonstration on the appearance of such symmetries, let us see how a \mathbf{Z}_2 discrete symmetry can arise [74]. Under a \mathbf{Z}_2 action, a point P on an elliptic curve is identified with its opposite, $-P$. From the group law, (see figure 2) $P + (-P) = O$, hence $P + P = O$, where for the Weierstraß form the zero element O is taken to infinity. This implies that line OP must be tangent to P . If we put P at the origin, $(0,0)$ then $dy/dx = \infty$. For example, the elliptic curve $y^2 = x(x^2 + x + 1)$ has a \mathbf{Z}_2 symmetry at the origin $(0,0)$.

4.1. On GUT Models with Mordell-Weil $U(1)$'s

In the geometric picture of F-theory discussed previously, the elliptic fibration assumed over a base B_3 can be defined as a holomorphic section of the fourfold. In the following, the possibility of having a fibration of an elliptic curve with two rational sections, including the zero (universal) section will be examined. This leads to a rank-one Mordell-Weil group, or a theory with one $U(1)$ symmetry in addition to the non-abelian GUT group. From the phenomenological point of view this is one of the most viable possibilities. GUT models with an additional abelian factor and perhaps a discrete symmetry arising from the torsion part (17), are probably adequate to impose sufficient constraints on superpotential terms.

However, in general, it is not easy to identify the $U(1)$ symmetries by starting directly from the Weierstraß form. Instead, it is more feasible to start with a different representation of the elliptic curve where the Mordell-Weil rank and other characteristics are more transparent. Once we have identified the abelian structure, in order to study the non-abelian group we convert our equation to the ordinary Weierstraß form using the appropriate birational transformation. To derive the equation of such a hypersurface, following the analysis of [52], we start with a point P associated to the holomorphic (zero) section and a rational point Q on an elliptic curve. We introduce the degree-two line bundle $\mathcal{M} = \mathcal{O}(P + Q)$ and denote u and v its two independent sections with weights $[1 : 1]$ generating the group $H^0(\mathcal{M})$. The space $H^0(2\mathcal{M})$ should have four independent sections. Given u and v we are able to generate only three, namely, u^2 , v^2 and uv . Thus we need to introduce a new one, let t with weight 2, so we are in a $\mathbf{P}_{(1,1,2)}$ projective space of three sections $[u, v, t]$ with weights $[1 : 1 : 2]$ respectively. From u, v, t we can form six sections of degree 6 (namely u^3, v^3, uv^2, u^2v and tu, tv) which match exactly the number of independent

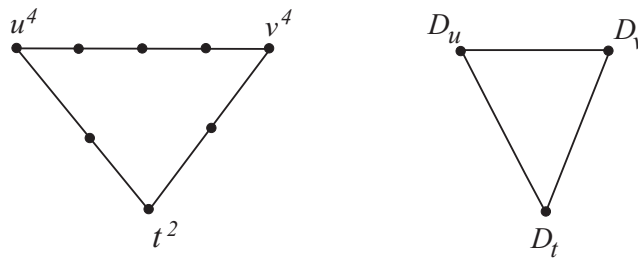


Figure 4. The reflexible polygon for the model $\mathbf{P}_{(1,1,2)}$ with one lattice point in its interior in accordance with the genus-one elliptic curve.

sections of $H^0(3\mathcal{M})$. But u, v, t generate nine sections for $H^0(4\mathcal{M})$ exceeding the independent ones by one. Hence there has to be a constraint among them which defines a hyper-surface in the weighted projective space $\mathbf{P}_{(1,1,2)}$ given by the equation of the form

$$t^2 + a_0 u^2 t + a_1 u v t + a_2 v^2 t = b_0 u^4 + b_1 u^3 v + b_2 u^2 v^2 + b_3 u v^3 + b_4 v^4 \quad (18)$$

where b_i, a_j are coefficients in the specific field \mathcal{K} we are interested in. The $\mathbf{P}_{(1,1,2)}$ projective space can be regarded as a toric variety [75]-[78] shown in the left side of figure 4. Furthermore, we can identify three divisors. For $t = v = 0, u \neq 0$ the divisor $D_u = [1 : 0 : 0]$, for $t = u = 0, v \neq 0$ the divisor $D_v = [0 : 1 : 0]$, and for $u = v = 0, t \neq 0$, the divisor $D_t = [0 : 0 : 1]$. These are indicated on the right side of the same figure. Without loss of generality [52] in order to avoid complications with square roots etc, we can simplify this equation to:

$$t^2 + a_2 v^2 t = u(b_0 u^3 + b_1 u^2 v + b_2 u v^2 + b_3 v^3) \quad (19)$$

Having constructed the elliptic curve equation with one Mordell-Weil $U(1)$, we would like now to transform this equation to the familiar $\mathbf{P}_{(2,3,1)}$ model. In fact this is inevitable; in order to identify the non-abelian part of the gauge symmetry, we need to read off the singularity structure from the coefficients in the Weierstraß form. It can be proved that the conversion can occur by two sets of equations [68] relating the sections of the $\mathbf{P}_{(1,1,2)}$ model to those of $\mathbf{P}_{(2,3,1)}$. Both transformations lead to equivalent results. The simplest one is [68]:

$$v = \frac{a_2 y}{b_3^2 u^2 - a_2^2 (b_2 u^2 + x)} \quad (20)$$

$$t = \frac{b_3 u y}{b_3^2 u^2 - a_2^2 (b_2 u^2 + x)} - \frac{x}{a_2} \quad (21)$$

$$u = z \quad (22)$$

We substitute the above to $\mathbf{P}_{(1,1,2)}$ model and we recover the Tate's form

$$\begin{aligned} y^2 + 2 \frac{b_3}{a_2} x y z \pm b_1 a_2 y z^3 &= x^3 \pm \left(b_2 - \frac{b_3^2}{a_2^2} \right) x^2 z^2 \\ &\quad - b_0 a_2^2 x z^4 - b_0 a_2^2 \left(b_2 - \frac{b_3^2}{a_2^2} \right) z^6 \end{aligned}$$

This is indeed in the desired $\mathbf{P}_{(2,3,1)}$ form however not all of the Tate's coefficients are independent. Comparing with the standard Tate's form given in (7) we observe that

$$\alpha_6 = \alpha_2 \alpha_4 \quad (23)$$

Type	Group	a_1	a_2	a_3	a_4	a_6	Δ
I_0	—	0	0	0	0	0	0
I_1	—	0	0	1	1	1	1
I_2^s	$SU(2)$	0	1	1	1	2	2
I_3^s	$SU(3)$	0	1	1	2	3	3
I_1^{*ns}	$SO(9)$	1	1	2	3	4	7
I_0^{*n}	$SO(8)^*$	1	1	2	2	3	6
IV^{*s}	\mathcal{E}_6	1	2	2	3	5	8
III^{*s}	\mathcal{E}_7	1	2	3	3	5	9

Table 2. Selected cases of Tate’s coefficients satisfying the relation $\alpha_6 = \alpha_2\alpha_4$. The Standard Model is naturally embedded in the exceptional groups only.

As we shall see in the following, this relation inevitably implies constraints on the non-abelian singularities. We restrict here the analysis in the Tate’s form of the Weierstraß equation since it is this form that we automatically obtain from the birational map. Hence, we assume the local expansion of the Tate’s coefficients which as a function of the “normal” coordinate they are given by (12) and (13).

To see the implications of the relation $\alpha_6 = \alpha_4\alpha_2$, we need to substitute in it the specific types of coefficients for each non-abelian group shown in Table 1.

We start the investigation with the $SU(n)$ singularities. According to Table 1 we must treat separately even $SU(2m)$ and odd $SU(2m+1)$ cases.

(i) $SU(2m)$ case. The vanishing orders of α_k ’s for $SU(2n)$ groups are

$$\alpha_2 = \alpha_{2,1}w, \quad \alpha_4 = \alpha_{4,m}w^m, \quad \alpha_6 = \alpha_{6,2m}w^{2m}$$

Substitution into (23) gives

$$\alpha_{2,1} \alpha_{4,m} w^{m+1} = \alpha_{6,2m} w^{2m}$$

which is satisfied only for $m = 1$, implying that only $SU(2)$ is compatible.

(ii) $SU(2m+1)$ case. Reading off the minimal powers of α_k ’s from Table 1, we get

$$\alpha_{2,1} \alpha_{4,m} w^{m+2} = \alpha_{6,2m} w^{2m+1}$$

This is also valid for $m = 1$, hence only $SU(3)$ is admissible.

Extending this analysis to the rest of the entries in Tate’s table, one finds that the most interesting cases arise for the exceptional groups. We observe that under the particular birational map to Tate’s form the only non-trivial admissible non-abelian singularities are \mathcal{E}_6 and \mathcal{E}_7 .

5. Conclusions

In this talk, we have described a variety of discrete symmetries in F-theory models emerging in various ways. Current F-theory constructions are based on the elliptically fibred internal space with \mathcal{E}_8 being the highest admissible singularity. The non-abelian part of the gauge symmetry of a particular effective GUT model emerges as a subgroup of the \mathcal{E}_8 . However, in low energy effective models additional discrete or continuous symmetries are required to suppress flavour

changing processes and prevent fast proton decay. To this end, two methods have been developed in the recent literature:

- A class of early F-theory effective models are fully embedded in \mathcal{E}_8 . As a consequence, the GUT symmetry is a subgroup of \mathcal{E}_8 while the commutant incorporates any additional symmetry of discrete or continuous nature which could be used to put constraints on the effective lagrangian.

- In recent works, abelian and discrete symmetries emerge from the a non-trivial Mordell-Weil group, i.e. the group of rational points of the elliptic curves associated to the fibration. In this approach, one constructs a representation of the elliptic curve with the desired rational sections and then finds the birationally equivalent Weierstraß equation.

In the presence of one Mordell-Weil factor in particular, we have shown that the birational transformations to Tate's form gives two viable gauge symmetries which are

$$\mathcal{E}_6 \times U(1)_{MW} \text{ and } \mathcal{E}_7 \times U(1)_{MW}$$

Although such a scenario looks rather restrictive, yet these exceptional groups contain all the well known GUTs, such as $SU(5)$, $SO(10)$ and the like, which can be readily obtained once we break the initial symmetry by a suitable mechanism, such as flux breaking, Wilson lines mechanism etc. Furthermore, as long as the rank-one Mordell-Weil is concerned, a novel generalisation of models in the context of elliptic fibrations has been proposed recently [73]. Finally, it should be pointed out that discrete symmetries appear naturally in F-theory compactifications without section and examples with $SU(5)$ GUT symmetry have been presented in [70]-[72].

We close our discussion with a few observations. The rather interesting fact in the procedure discussed in the last section, is that the $U(1)_{MW}$ symmetry is not necessarily identified with some generator of the Cartan subalgebra of \mathcal{E}_8 . This means that the $U(1)_{MW}$ charges of the non-abelian representations are not necessarily the usual ones. Furthermore, the torsion group has a rich structure of discrete symmetries which can also be symmetries of the effective lagrangian. Perhaps issues such as proton stability, the μ -term and flavour physics find their solutions in a judicious choice of these symmetries.

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