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Analytical solutions to the (3+1)-dimensional generalized nonlinear Schrödinger equation with varying parameters

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Abstract

Analytical solutions to the (3+1)-dimensional nonlinear Schrödinger equation (NLSE) are obtained. It is the first time to our knowledge that analytical solutions to the three-dimensional NLSE with varying coefficients have been derived. The solutions are the generalization of analytical solutions to the one-dimensional NLSE. The properties of the amplitudes and the phases of the solutions are investigated.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The analytical solution to the one-dimensional (1D) normalized nonlinear Schrödinger equation [1, 2] in nonlinear optics (or the Gross–Pitaevskii equation (GPE) in the context of Bose–Einstein condensate [3]) $\mathrm{i}\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u^2|u = 0$, has been obtained by various kinds of methods, such as inverse scattering transformation [4], Darboux–Bäcklund transformation [5], self-similarity technique [6] and variational method [7] as well as the moment method [8]. The analytical soliton solution is in the form of the traveling wave hyperbolic secant function $u = \eta\ \mathrm{sech}\ \eta(t + vx + \theta_0)\,\mathrm{e}^{\mathrm{i}(-kt+\frac{1}{2}(\eta^2-v^2)x+\varphi_0)}$, where η and v represent the amplitude and the velocity of the soliton. However, no analytical solutions to the higher-dimensional NLSE have been obtained, to the best of our knowledge. In this paper, we study the (3+1)-dimensional (3D) generalized nonlinear Schrödinger equation (GNLSE) in the presence of gain (loss) γ

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and a harmonic potential $\frac{1}{2}s\rho^2$; most importantly, all the coefficients are variable along with evolution time t:

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{1}{2}\beta(t)\Delta u+\chi(t)\,|u|^2\,u-\mathrm{i}\gamma(t)u+\frac{1}{2}s(t)\rho^2u=0, \tag{1}$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the Laplace operator, $\rho^2=x^2+y^2+z^2$, and β and χ are the diffractive (or dispersive) coefficient and nonlinear coefficient, respectively. In the context of Bose–Einstein condensate, equation (1) is also called the Gross–Pitaevskii equation which governs the evolution of matter waves. The NLSE (GPE) model is a universal nonlinear model that appears in many branches of physics and applied mathematics, including nonlinear optics [9, 10], condensed matter and plasma physics [11–13], nonlinear quantum field theory, fluid mechanics, etc. While 1D NLSE with varying parameters is always utilized to describe the problems of dispersion and nonlinearity management [14, 15], few NLSEs with higher dimensions and with varying parameters have been investigated.

2. Analytical solutions to the 3D GNLSE

Our strategy to solve partial equation (1) is as follows: considering that the 1D NLSE possesses the traveling wave hyperbolic secant function solution, and the hyperbolic secant function is a special case of the Jacobi elliptic function, we search for the traveling wave Jacobi elliptic function solution to the 3D GNLSE. And a vortex phase $a(x^2+y^2+z^2)$ is set to reflect a rotation-invariable feature of the 3D GNLSE. According to the strategy, we look for the solution to the 3D GNLSE (1) in the form

$$u(t,x,y,z)=A(t,x,y,z)\,\mathrm{e}^{\mathrm{i}B(t,x,y,z)}, \tag{2}$$

where $A=f_0(t)+f_1(t)F(\theta)+f_{-1}(\theta)F^{-1}(\theta)$ and $B=a(t)\rho^2+b(t)(x+y+z)+c(t)$. $F(\theta)$ is a Jacobi elliptic function which is the solution of the ordinary differential equation $(F')^2=q_0+q_2F^2+q_4F^4$, $\theta(t,x,y,z)=k(t)x+l(t)y+p(t)z+w(t)t$ is the traveling wave variable, $F^{-1}(\theta)=1/F(\theta)$, and by incorporating t into $w(t)$, θ is rewritten as $\theta(t,x,y,z)=k(t)x+l(t)y+p(t)z+w(t)$. a is the curvature of the wave-front, the spatial frequency shift b is relative to the parallel motion of the wave-front and c is the homogeneous phase shift. All the coefficients are time dependent.

Substituting equations (2) into (1), we get the coupling equations:

$$\frac{\partial A}{\partial t}+\beta\left[\frac{\partial A}{\partial x}\frac{\partial B}{\partial x}+\frac{\partial A}{\partial y}\frac{\partial B}{\partial y}+\frac{\partial A}{\partial z}\frac{\partial B}{\partial z}+\frac{A}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)B\right]-\gamma A=0, \tag{3}$$

$$-A\frac{\partial B}{\partial t}+\frac{\beta}{2}\left[\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+\frac{\partial^2 A}{\partial z^2}-A\left(\frac{\partial B}{\partial x}\right)^2-A\left(\frac{\partial B}{\partial y}\right)^2-A\left(\frac{\partial B}{\partial z}\right)^2\right]$$
$$+\chi A^3+\frac{1}{2}s(x^2+y^2+z^2)A=0. \tag{4}$$

Further substituting the expressions of A, B, θ into (3) and (4), utilizing the properties of the elliptic function and imposing the coefficients of x^qF^n, y^qF^n $(q=0,1,2;\quad n=0,1,2,3)$ (F' being separately equal to zero), we obtain a system of algebraic or first-order ordinary differential equations for f_i, k, l, ω, a, b and c, from which we obtain the results of the coefficients:

$$f_0=0;\quad f_i=f_{i0}\,\mathrm{e}^{\int_0^t(\gamma-3a\beta)\,\mathrm{d}t'}\quad (i=1,-1), \tag{5a}$$

$$w=-\int_0^t\beta b(k+l+p)\,\mathrm{d}t'+w_0, \tag{5b}$$

Table 1. Relations between the values of (q_0, q_2, q_4) and corresponding $F(\xi)$ in the ordinary differential equation $F'^2 = q_0 + q_2F^2 + q_4F^4$

q_0	q_2	q_4	$F'^2 = q_0 + q_2F^2 + q_4F^4$	F
1	$-(1+m^2)$	m^2	$F'^2 = (1-F^2)(1-m^2F^2)$	$\mathrm{sn}\,\xi,\ \mathrm{cd}\,\xi = \frac{\mathrm{cn}\,\xi}{\mathrm{dn}\,\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$F'^2 = (1-F^2)(m^2F^2+1-m^2)$	$\mathrm{cn}\,\xi$
m^2-1	$2-m^2$	-1	$F'^2 = (1-F^2)(F^2+m^2-1)$	$\mathrm{dn}\,\xi$
m^2	$-(1+m^2)$	1	$F'^2 = (1-F^2)(m^2-F^2)$	$\mathrm{ns}\,\xi = (\mathrm{sn}\,\xi)^{-1}$, $\mathrm{dc}\,\xi = \frac{\mathrm{dn}\,\xi}{\mathrm{cn}\,\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$F'^2 = (1-F^2)[(m^2-1)F^2-m^2]$	$\mathrm{nc}\,\xi = (\mathrm{cn}\,\xi)^{-1}$
-1	$2-m^2$	m^2-1	$F'^2 = (1-F^2)[(1-m^2)F^2-1]$	$\mathrm{nd}\,\xi = (\mathrm{dn}\,\xi)^{-1}$
1	$2-m^2$	$1-m^2$	$F'^2 = (1+F^2)[(1-m^2)F^2+1]$	$\mathrm{sc}\,\xi = \frac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$F'^2 = (1+m^2F^2)[1+(m^2-1)F^2]$	$\mathrm{sd}\,\xi = \frac{\mathrm{sn}\,\xi}{\mathrm{dn}\,\xi}$
$1-m^2$	$2-m^2$	1	$F'^2 = (1+F^2)(F^2+1-m^2)$	$\mathrm{cs}\,\xi = \frac{\mathrm{cn}\,\xi}{\mathrm{sn}\,\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$F'^2 = (F^2+m^2)(F^2+m^2-1)$	$\mathrm{ds}\,\xi = \frac{\mathrm{dn}\,\xi}{\mathrm{sn}\,\xi}$

$$k = k_0\,\mathrm{e}^{-2\int_0^t \beta a \mathrm{d}t'}, \quad l = l_0\,\mathrm{e}^{-2\int_0^t \beta a \mathrm{d}t'}, \quad p = p_0\,\mathrm{e}^{-2\int_0^t \beta a \mathrm{d}t'}, \tag{5c}$$

$$c = \left[\int_0^t \frac{\beta}{2}(k^2+l^2+p^2)q_2 - \frac{3\beta}{2}b^2 + 3\chi f_1 f_{-1}\right]\mathrm{d}t' + c_0, \tag{5d}$$

$$b = b_0\,\mathrm{e}^{-2\int_0^t \beta a \mathrm{d}t'}, \tag{5e}$$

$$a' = -2\beta a^2 + \frac{s}{2}, \tag{5f}$$

$$\chi = -\frac{k^2+l^2+p^2}{f_1^2}\beta q_4 \quad (\text{when } f_1 \neq 0) \quad \text{or} \quad \chi = -\frac{k^2+l^2+p^2}{f_{-1}^2}\beta q_0 \quad (\text{when } f_{-1} \neq 0) \tag{5g}$$

$$f_{-1} = \pm\sqrt{\frac{q_0}{q_4}}f_1 \qquad (\text{when } f_1 \neq 0 \text{ and } f_{-1} \neq 0). \tag{5h}$$

What is amazing and delectable is that all the formulas are self-consistent and self-contained, which means that our solving procedure is successful.

The only necessary and sufficient condition for the existence of the Jacobi elliptic function solution to the 3D GNLSE is that the nonlinearity and the dispersion should fulfill a certain relation expressed by equation (5*g*). This condition is actually a generalized constraint for the generalized NLSE to have elliptic function solutions including the hypersecant and hypertangent soliton solutions.

As for the 1D NLSE with varying parameters, solitons exist only under certain conditions [14, 15]. From equation (5) of [14], the necessary condition for the 1D GNLSE to have soliton solutions is $\chi = c\beta$, where c is a constant, and equation (1) in [14] can be normalized so as that $\chi = \pm\beta$. In our paper, constraint (5*g*) results in $\chi = -q_4\beta$ assuming that $k_0 = 1$, $l_0 = p_0 = 0$, $f_{10} = 1$, $\gamma = 0$ and $a = 0$. As for elliptic functions, $\mathrm{cn}\,\xi \to \mathrm{sech}\,\xi$ when $m = 1$ and then $q_4 = -1$ (see table 1) and $\chi = \beta$ is obtained in this case; $\mathrm{sn}\,\xi \to \tanh\xi$ when $m = 1$ and then $q_4 = 1$ and $\chi = -\beta$. This is consistent with the above special case in [15] as well as the 1D normalized equation in the introduction with constant coefficients.

So the condition for the existence of the soliton solutions of the 1D NLSE is just a special case of the constraint in our paper. This constraint can be realized in some situations, especially in the case of constant coefficients. This constraint applies to the physically relevant systems apart from the cases when $\gamma = s = a = 0$.

Substituting the resulting expressions (5*a*)–(5*h*) into equation (2), we get the following analytical solutions (6) and (7) to the 3D GNLSE. Solutions (6) and (7) are the single elliptic function and hybrid elliptic function solution, respectively:

$$u(t,x,y,z) = f_{10}\,\mathrm{e}^{\int_0^t(\gamma-3a\beta)\mathrm{d}t'}\,F(\theta)\,\mathrm{e}^{\mathrm{i}[a(x^2+y^2+z^2)+b(x+y+z)+c]} \qquad (f_{-1}=0), \tag{6}$$

or

$$u(t,x,y,z) = f_{10}\,\mathrm{e}^{\int_0^t(\gamma-3a\beta)\mathrm{d}t'}\left[F(\theta)\pm\sqrt{\frac{q_0}{q_4}}F^{-1}(\theta)\right]\mathrm{e}^{\mathrm{i}[a(x^2+y^2+z^2)+b(x+y+z)+c]} \qquad (f_1\neq 0 \text{ and } f_{-1}\neq 0), \tag{7}$$

where a is determined by equation (5*f*) and the initial condition, $\theta = (k_0x+l_0y+p_0z)\,\mathrm{e}^{-2\int_0^t\beta a\mathrm{d}t'} - \int_0^t\beta b_0(k_0+l_0+p_0)\,\mathrm{e}^{-4\int_0^{t'}\beta a\mathrm{d}t''}\mathrm{d}t' + w_0$, $b = b_0\,\mathrm{e}^{-2\int_0^t\beta a\mathrm{d}t'}$, and $c = \left[\frac{q_2}{2}\left(k_0^2+l_0^2+p_0^2\right)-\frac{3}{2}b_0^2\right]\int_0^t\beta\,\mathrm{e}^{-4\int_0^{t'}\beta a\mathrm{d}t''}\mathrm{d}t' + c_0$ in equation (6) or $c = \left[\frac{q_2}{2}\left(k_0^2+l_0^2+p_0^2\right)-\frac{3}{2}b_0^2 \mp 3\sqrt{q_0q_4}\left(k_0^2+l_0^2+p_0^2\right)\right]\int_0^t\beta\,\mathrm{e}^{-4\int_0^{t'}\beta a\mathrm{d}t''}\mathrm{d}t' + c_0$ in equation (7).

When $s = 0$, the solutions can be greatly simplified. In this case, $a = \frac{a_0}{1+2a_0\int_0^t\beta\mathrm{d}t'} = \varepsilon a_0$, where $\varepsilon = \left(1+2a_0\int_0^t\beta\mathrm{d}t'\right)^{-1}$. When $a_0\neq 0$, the solutions are respectively

$$u = f_{1,0}\varepsilon^{3/2}\,\mathrm{e}^{\int_0^t\gamma\mathrm{d}t'}\,F(\theta)\,\mathrm{e}^{\mathrm{i}[a(x^2+y^2+z^2)+b(x+y+z)+c]}, \tag{6a}$$

and

$$u = f_{1,0}\varepsilon^{3/2}\,\mathrm{e}^{\int_0^t\gamma\mathrm{d}t'}\left[F(\theta)\pm\sqrt{\frac{q_0}{q_4}}F^{-1}(\theta)\right]\mathrm{e}^{\mathrm{i}[a(x^2+y^2+z^2)+b(x+y+z)+c]}, \tag{7a}$$

where $\theta = \varepsilon(k_0x+l_0y+p_0z)+\frac{b_0}{2a_0}(k_0+l_0+p_0)(\varepsilon-1)+w_0$, $b=\varepsilon b_0$ and $c = \frac{1-\varepsilon}{4a_0}\left[\left(k_0^2+l_0^2+p_0^2\right)q_2-3b_0^2\right]+c_0$ in equation (6*a*) or $c = \frac{1-\varepsilon}{4a_0}\left[\left(k_0^2+l_0^2+p_0^2\right)q_2-3b_0^2\mp 6\sqrt{q_0q_4}\left(k_0^2+l_0^2+p_0^2\right)\right]+c_0$ in (7*a*). If the initial curvature of the wave-front is $a_0=0$, the solutions are

$$u(t,x,y,z) = f_{10}\,\mathrm{e}^{\int_0^t\gamma\mathrm{d}t'}\,F(\theta)\,\mathrm{e}^{\mathrm{i}[b(x+y+z)+c]}, \tag{6b}$$

or

$$u(t,x,y,z) = f_{10}\,\mathrm{e}^{\int_0^t\gamma\mathrm{d}t'}\left[F(\theta)\pm\sqrt{\frac{q_0}{q_4}}F^{-1}(\theta)\right]\mathrm{e}^{\mathrm{i}[b(x+y+z)+c]}, \tag{7b}$$

where $b=b_0$, $\theta = k_0x+l_0y+p_0z-b_0(k_0+l_0+p_0)\int_0^t\beta\mathrm{d}t'+w_0$, $c = \left[\frac{q_2}{2}\left(k_0^2+l_0^2+p_0^2\right)-\frac{3}{2}b_0^2\right]\int_0^t\beta\mathrm{d}t'+c_0$ in equation (6*b*) or $c = \left[\frac{q_2}{2}\left(k_0^2+l_0^2+p_0^2\right)-\frac{3}{2}b_0^2\mp 3\sqrt{q_0q_4}\left(k_0^2+l_0^2+p_0^2\right)\right]\int_0^t\beta\mathrm{d}t'+c_0$ in equation (7*b*).

3. Analyses and discussions

In order to manifest the solution, we substitute equation (6) into equation (1), where we set the following analytical parameters: $\gamma = 0.1\ \cos t$, $\beta = \frac{3\cos t}{2(2\sin t+4t+1)}$, $s = \frac{4(2+\cos t)}{3(2+\sin t)}$, so that the explicit wave-front curvature $a = \frac{2\sin t+4t+1}{3\sin t+6}$ and the spatial frequency shift $b = \frac{2}{2+\sin t}$ are derived. The nonlinear coefficient is $\chi = \frac{k_0^2+l_0^2+p_0^2}{f_{10}^2}q_4\frac{3\cos t(2+\sin t)}{4(2\sin t+4t+1)}\,\mathrm{e}^{-0.2\sin t}$ according to the

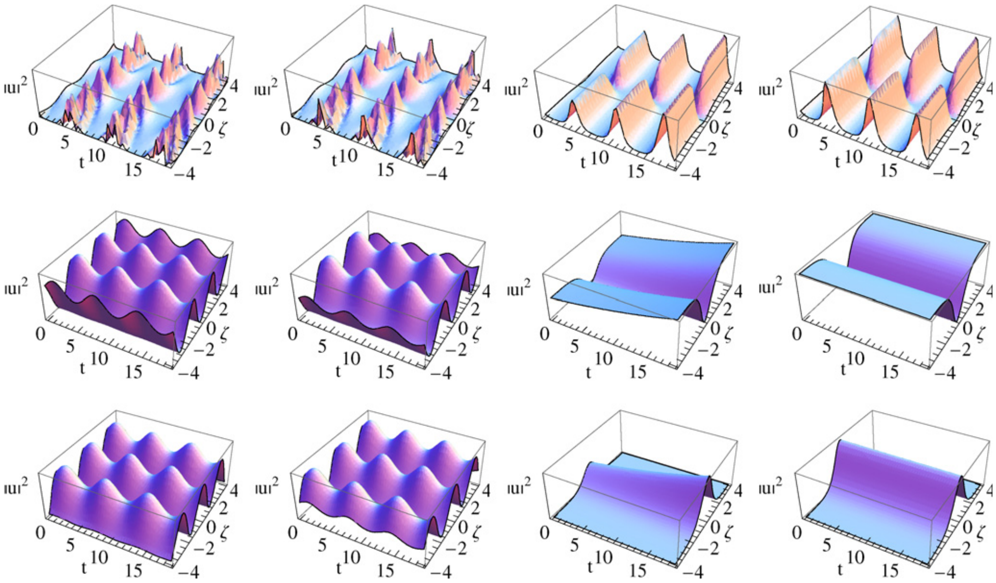

Figure 1. 3D intensity $|u(t,x)|^2$ plots of Jacobian sn (rows 1 and 2) and Jacobian cn function (row 3) solutions. The varying parameters are set as $\beta = \frac{3\cos t}{2(2\sin t+4t+1)}$, $s = \frac{4(2+\cos t)}{3(2+\sin t)}$ and $a_0 = \frac{1}{6}$ in row 1; $\beta = \frac{3\cos t}{2(2\sin t+4t+1)}$, $s = 0$ and $a = 0$ in rows 2 and 3; $\gamma = 0.1\cos t$ in columns 1 and 2, $\gamma = 0.02$ in column 3 and $\gamma = 0$ in column 4. The first two columns correspond to $m = -0.5$ and $m = 0$, respectively, and the last two columns correspond to $m = 1$. The label of the spatial coordinate axis is $\xi = k_0x + l_0y + p_0z$.

constraint (5*g*). We find that equation (1) is satisfied, which proves that equation (6) is just the solution to equation (1). It should be pointed out that the functions γ, β and s are chosen ad hoc in order to get analytical expressions for the parameters a, b and χ, and to simplify the calculation.

There are 12 different Jacobi elliptic functions, each of which is differentiated by different order m (see table 1). Jacobi elliptic functions degenerate into triangular functions when $m = 0$, and hyperbolic functions when $m = 1$. The amplitude of the light wave or matter wave should be finite; thus, the elliptic functions that end with 's' (ns, cs, ds) cannot be selected to be the amplitude function. All that end with 'n' (sn, cn, dn) can absolutely be the amplitude function in any order, and all that end with 'd' or 'c' can be the amplitude function in some orders. As for the hybrid solution, only $F(\theta) = \mathrm{cn}(\theta,m) \pm \sqrt{\frac{m^2-1}{m^2}}\mathrm{nc}(\theta,m)(m > 1)$ and $F(\theta) = \mathrm{dn}(\theta,m) \pm \sqrt{1-m^2}\mathrm{nd}(\theta,m)r\ (-1 < m < 1)$ can be wavefunctions.

Depicted in figure 1 are the wave intensities of Jacobi sn (rows 1 and 2) and Jacobi cn solution (row 3). Without losing generality, the initial conditions are set to be $b_0 = 1$, $w_0 = c_0 = 0$, $f_{10} = 1$. The varying parameters are still set as $\beta = \frac{3\cos t}{2(2\sin t+4t+1)}$, $s = \frac{4(2+\cos t)}{3(2+\sin t)}$ and $a_0 = \frac{1}{6}$ in row 1, which depicts the general case; $\beta = \frac{3\cos t}{2(2\sin t+4t+1)}$, $s = 0$ and $a = 0$ in rows 2 and 3. We set $\gamma = 0.1\cos t$ in columns 1 and 2, $\gamma = 0.02$ in column 3 and $\gamma = 0$ in column 4. The first two columns correspond to $m = -0.5$ and $m = 0$, respectively, and the last two columns correspond to $m = 1$.

The conventional one-dimensional hyperbolic secant (hyperbolic tangent) soliton [16, 17] solution is the special case of our results (see the last figures in rows 2 and 3). In these cases, χ and β must be of the same sign for hypersecant (bright) soliton and of the contrary sign

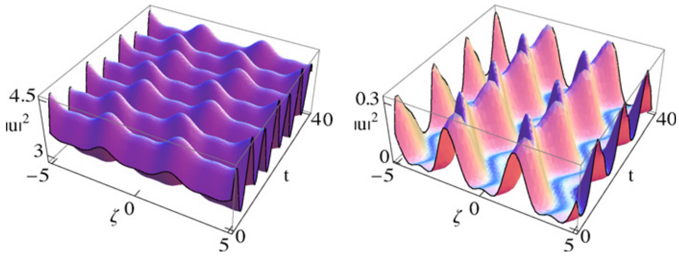

Figure 2. 3D intensity $|u(t,x)|^2$ of Hybrid Jacobi elliptic function solutions, where $F(\theta) = \mathrm{dn}(\theta, m) + \sqrt{1-m^2}\mathrm{nd}(\theta, m)$ (left) and $F(\theta) = \mathrm{dn}(\theta, m) - \sqrt{1-m^2}\mathrm{nd}(\theta, m)$ (right). The parameters are set to be $m = 0.5$, $\gamma = 0.5\cos t$, $\beta = \cos 0.5t$, $s = 0$ and $a = 0$. The label of the spatial coordinate axis is $\xi = k_0x + l_0y + p_0z$.

for the hypertangent (black) soliton solution which derived from the necessary condition (5g). The periodical gain (loss) periodically amplifies (deduces) the amplitude of the intensity, and net gain (loss) amplifies (deduces) the amplitude.

In the case of planar wave-front (i.e. $a = 0$), the energy conserves provided that there is no gain (loss) (see the lower two lines in figure 1); however, in the case of spherical wave-front (i.e. $a \neq 0$), the energy fluctuates along with time even if there is no gain (loss), as shown in the upper row in figure 1. This phenomenon has never been found before. We think that the nonconservation of the energy is mathematically due to the presence of the wave-front curvature, and physically due to the presence of the potential. When $s \neq 0$, the wave-front a will never be zero, which can derived from equation (5f). Hence, the coefficient of the elliptic function in equation (7) will vary along with time t, which results in the nonconservation of energy. In fact, an adscititious potential of course changes the total energy of the system.

Figure 2 depicts the hybrid Jacobi elliptic function solutions $F(\theta) = \mathrm{dn}(\theta, m) + \sqrt{1-m^2}\mathrm{nd}(\theta, m)$ (left) and $F(\theta) = \mathrm{dn}(\theta, m) - \sqrt{1-m^2}\mathrm{nd}(\theta, m)$ (right), where the parameters are set to be $m = 0.5$, $\gamma = 0.5\cos t$, $\beta = \cos 0.5t$, $s = 0$ and $a = 0$. These solutions have never been obtained before, and they are absolutely different from the single elliptic function solutions.

When the potential is absent, and an initial wave with a wave-front curvature ($a_0 \neq 0$) is input, the curvature of the phase wave-front evolves in the form of $a = \frac{a_0}{1+2a_0\int_0^t \beta \mathrm{d}t'}$, but the curvature center fixes at $\left(-\frac{b_0}{2a_0}, -\frac{b_0}{2a_0}, -\frac{b_0}{2a_0}\right)$. If the initial wave-front is a planar one ($a_0 = 0$), it remains a planar wave-front. The presence of the potential complicates the phase of the solution. Even if the initial wave-front is a planar one, it will evolve into a spherical wave-front, and both the curvature a and its center $\left(-\frac{b}{2a}, -\frac{b}{2a}, -\frac{b}{2a}\right)$ will vary along with time, where a and b are determined by equations (5f) and (5e), respectively.

4. Conclusions

We obtain analytical solutions to the 3D GNLSE with time-dependent coefficients for the first time to our knowledge. The solutions are in the form of the traveling wave elliptic functions, which are the generalizations of the conventional hyperbolic function soliton solutions to the 1D NLSE. The method that we use to solve the NLSE will pave the way to new methods for solving high-dimensional partial differential equations.

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