PAPER

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IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **49** (2016) 335401 (15pp) doi:10.1088/1751-8113/49/33/335401

A note on the loop formula for the fermionic determinant

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Received 12 January 2016, revised 6 May 2016
Accepted for publication 7 June 2016
Published 8 July 2016

Abstract

A formula expressing the fermionic determinant as an infinite product of smaller determinants is derived and discussed. These smaller determinants are of a fixed size, independent of the size of the lattice and are indexed by loops of increasing length.

Keywords: lattice fermions, determinant, loops

1. Introduction

The study of the effects of virtual particles has a very long history. In particular the vacuum polarization due to electron–positron pairs was studied first by Euler and Heisenberg [1] and later by Schwinger [2].

Let us mention some of the later developments: the vacuum polarization to all orders is given by the fermion determinant. Its mathematical properties were studied later for instance by Seiler [3] in the context of constructive quantum field theory, where a stability problem arising in four dimensional quantum electrodynamics was pointed out. This latter issue was studied more extensively by [4]. The study of this issue culminated in the work of Fry [7] which also includes the effect of fermions on non-abelian gauge fields.

The quantum effects on vortex fields were analyzed by Langfeld *et al* [5], using Schwinger's propert time formalism. Schmidt and Stamatescu [6], also using this formalism, pointed out that the fermion (and boson) determinant on the lattice can be viewed as a gas of closed loops which can be simulated numerically via a random walk.

In this note we also consider lattice gauge theories; we derive and discuss a general loop formula for the fermion determinant. This formula provides a systematic approximation for the fermionic determinants which can be used in full QCD analyses. In particular it proved

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useful in problems of QCD at non-zero chemical potential where it allowed to set up the so called HD-QCD (for high density quantum chromodynamics) approximation for large mass and chemical potential [8–10] which was used for physically relevant simulations in leading (LO) and next-to-leading orders (NLOs) [11–14]. This formula involves, however, some subtleties both in its derivation and in its interpretation and we think it is of some interest to address them here.

Let us repeat that the formula itself is not new, but in this note we provide some necessary clarifictions of its status.

This loop formula is based on the loop expansion derived long ago by one of us [15] for the logarithm of the determinant with Wilson fermions in an external gauge field. This logarithm is expressed there as a power series expansion in the hopping parameter κ, in which each term is given as a sum of closed loops. The loop formula proceeds from this expansion and expresses the full determinant as a product over determinants of a much smaller fixed dimension (the tensor product of spin and color spaces) involving closed loops on the lattice of increasing complexity [12]. It involves the resummation of nested infinite series and therefore its correctness beyond formal algebra depends on (absolute) convergence. We will discuss in more detail the derivation of the formula as well as its limitations and possible misunderstandings in using it, since one might think that the zeroes of the determinant are given by the zeroes of the factors of the product. In fact this is not the case, as we will explain in the following, and the formula should instead be understood as a systematic approximation, as illustrated below.

In particular for a finite lattice this formula expresses the determinant which is a polynomial of finite order in the hopping parameter as an infinite product. Obviously this can make sense only where the infinite product converges, which is equivalent to convergence of the expansion of its logarithm; this convergence will break down at the latest at the first zero encountered, either on the left-hand side (lhs) or in one of the factors on the right-hand side (rhs).

A simple example might help at this point to illustrate both derivation and problems of the loop formula. Consider

$$1 - \kappa(x + y) = \mathrm{e}^{\ln(1-\kappa(x+y))}. \tag{1}$$

We expand the logarithm in powers of κ and x, y, but for bookkeeping purposes we treat x, y as non-commuting symbols, so we consider the string $xxyy$ as different from $xyxy$. This way we get

$$\begin{aligned}\ln(1 - \kappa(x + y)) = &- \kappa x - \kappa y - \frac{\kappa^2}{2}(xx + yy + 2xy)) \\ &- \frac{\kappa^3}{3}(xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy) \\ &- \frac{\kappa^4}{4}(xxxx + xxxy + \cdots) + \cdots.\end{aligned} \tag{2}$$

Symbolically we can write

$$\ln(1 - \kappa(x + y)) = -\sum_l \frac{\kappa^l}{l} \sum_{s_l} s_l(x, y), \tag{3}$$

where $s_l(x, y)$ stands for strings of lengh l formed from x and y. Next we introduce 'primary strings' s_l^P as those strings that are not repetitions of other strings and resum now first over all repetitions of primary strings ('s-resummation' in the following) and then over the primary

strings. This way we get

$$\ln(1 - \kappa(x + y)) = -\sum_{l} \sum_{s_l^P} \sum_{s} \frac{\kappa^{ls}}{ls} s_l^P(x, y)^s. \tag{4}$$

Up to cyclic permutations the primary strings of length 1 are x and y, of length 2 there is only xy, of length 3 there are xxy, yyx, xyy, yxx, xyx, yxy, of length 4 $xxxy$, $xxyy$ etc. Notice that $xyxy$ is not a primary string but a repetition of the string xy. We now introduce equivalence classes $[s_l^P]$ for the l strings of length l differing by cyclic permutations and obtain

$$\begin{aligned}\ln(1 - \kappa(x + y)) &= -\sum_{l} \sum_{[s_l^P]} \sum_{s} \frac{\kappa^{ls}}{s} s_l^P(x, y)^s \\ &= \sum_{l} \sum_{[s_l^P]} \ln(1 - \kappa^l s_l^P(x, y)).\end{aligned} \tag{5}$$

So we obtain

$$\begin{aligned}\ln(1 - \kappa(x + y)) = {} & \ln(1 - \kappa x) + \ln(1 - \kappa y) + \ln(1 - \kappa^2 xy) \\ & + \ln(1 - \kappa^3 x^2 y) + \ln(1 - \kappa^3 x y^2) + \ln(1 - \kappa^4 x^3 y) \\ & + \ln(1 - \kappa^4 x y^3) + \ln(1 - \kappa^4 x^2 y^2) + \cdots.\end{aligned} \tag{6}$$

After exponentiation this gives

$$\begin{aligned}& 1 - \kappa(x + y) \\ & \quad = (1 - \kappa x)(1 - \kappa y)(1 - \kappa^2 xy)(1 - \kappa^3 x^2 y)(1 - \kappa^3 x y^2) \\ & \qquad \times (1 - \kappa^4 x^2 y^2)(1 - \kappa^4 x^3 y)(1 - \kappa^4 x y^3) \times \cdots.\end{aligned} \tag{7}$$

Regarding now the symbols x, y again as complex numbers we turn to the question of convergence. One can see in this example that truncating the product by keeping only the factors containing up to κ^4 the lhs is only reproduced up to order κ^5:

$$\begin{aligned}& (1 - \kappa x)(1 - \kappa y)(1 - \kappa^2 xy)(1 - \kappa^3 x^2 y)(1 - \kappa^3 x y^2) \\ & \quad \times (1 - \kappa^4 x^2 y^2)(1 - \kappa^4 x^3 y)(1 - \kappa^4 x y^3) \\ & \quad = 1 - \kappa(x + y) + \kappa^5(x^4 y + 2x^3 y^2 + 2x^2 y^3 + x y^4) + O(\kappa^6).\end{aligned} \tag{8}$$

Of course this mechanism will operate the same way at any order: truncation at order κ^n will produce an approximation to the determinant up to order $O(\kappa^{n+1})$.

To determine when this expansion converges, note that for $x, y \geqslant 0$ all terms in equation (2) have the same sign, so this expansion converges absolutely for $\kappa < 1/(x + y)$. For general x, y we thus have absolute convergence of any reordering and resumming of the expansion for $\ln(1 - \kappa(x + y))$, provided

$$\kappa < \frac{1}{|x| + |y|}, \tag{9}$$

so in particular the expansion equation (5) converges. Exponentiating we obbtain convergence of the infinite product equation (7).

One can see directly that in this region neither the lhs nor any of the factors on the rhs vanishes, so in the region of convergence there is no paradox of one side vanishing with the other side non-vanishing. The infinite product on the rhs converges in this region to the lhs, which provides the analytic continuation to all of $\mathbb{C}^2$.

It should not be surprising that the location of the zeroes of the lhs is not well approximated by the zeroes of the factors on the rhs: the lowest zeroes as given by the two

linear factors, $(1 - \kappa x)(1 - \kappa y)$, e.g., are at $\kappa = 1/x$ and $\kappa = 1/y$, whereas the true zero is at $\kappa = 1/(x + y)$. Only after expanding the truncated product we obtain a first order approximation of the lhs: $(1 - \kappa x)(1 - \kappa y) = 1 - \kappa(x + y) + O(\kappa^2)$.

In the following we shall first present the principle leading to this formula in section 2, then briefly describe its derivation for QCD with Wilson fermions, which involves some subtleties, in section 3. Details and illustration of both the derivation of the loop formula and of its problems are given in the appendices.

2. The idea in a nutshell

The principle behind the formula is easy to explain: let

$$W = \mathbb{1} - \kappa M; \quad M_{ii} = 0, \ \ i = 1, \cdots n \tag{10}$$

be a $N \times N$ matrix. Then

$$\ln \det W = -\sum_{l=1}^{\infty} \frac{\kappa^l}{l} \mathrm{tr}\, M^l \tag{11}$$

which converges as long as $||M|| < 1/\kappa$ or even as long as the spectral radius $\rho(M) < 1$. We can interprete $\mathrm{tr}M^l$, which involves a l-fold sum over the matrix indices as a sum over all closed paths (journeys or loops) of length l over the the index set. To be precise, we define a path of length l as a map

$$\mathcal{C}_l : \{0, 1, \cdots, l\} \to \{1, 2, \cdots N\} \tag{12}$$

with $\mathcal{C}_l(l) = \mathcal{C}_l(0)$. Furthermore we define a weight for each path

$$M_{\mathcal{C}_l} = \prod_{i=0}^{l-1} M_{\mathcal{C}_l(i),\mathcal{C}_l(i+1)}, \tag{13}$$

so that

$$\mathrm{tr}M^l = \sum_{\mathcal{C}_l} M_{\mathcal{C}_l}. \tag{14}$$

and

$$\ln \det W = -\sum_{l=1}^{\infty} \sum_{\mathcal{C}_l} \frac{\kappa^l}{l} M_{\mathcal{C}_l}. \tag{15}$$

The weight defined in equation (13) is clearly invariant under cyclic permutations of $\{0, 1, \cdots, l\}$, so it makes sense to introduce equivalence classes $[\mathcal{C}_l]$ of paths $\mathcal{C}_l$ that can be mapped onto each other by a cyclic permutation. So we finally can write

$$\ln \det W = -\sum_{l=1}^{\infty} \sum_{[\mathcal{C}_l]} \kappa^l M_{\mathcal{C}_l}. \tag{16}$$

As long as the series (15) converges absolutely, we can make any kind of rearrangement, summing first over (finite or infinite) subsets of paths; the result will not be affected. For instance we can define *primary paths* $\mathcal{C}_l^P$ of length l as paths which are not repetitions of other paths, and rewrite the formula as

$$\ln \det W = -\sum_{l=1}^{\infty} \sum_{\mathcal{C}_l^P} \sum_{s=1}^{\infty} \frac{\kappa^{sl}}{sl} (M_{\mathcal{C}_l^P})^s, \tag{17}$$

$$= -\sum_{l=1}^{\infty} \sum_{[\mathcal{C}_l^P]} \sum_{s=1}^{\infty} \frac{\kappa^{sl}}{s} (M_{\mathcal{C}_l^P})^s = -\sum_{l=1}^{\infty} \sum_{[\mathcal{C}_l^P]} \ln(1 - \kappa^l M_{\mathcal{C}_l^P}). \tag{18}$$

After exponentiating, this gives

$$\det W = \prod_{l=1}^{\infty} \prod_{[\mathcal{C}_l^P]} (1 - \kappa^l M_{\mathcal{C}_l^P}). \tag{19}$$

This last expression looks like a factorization of $\det W$, but it is not, because the convergence requirement limits its validity. Absolute convergence can hold at most for $|\kappa| < |\kappa_c|$ where κ_c is the zero of $\det W$ closest to the origin. But of course it may break down earlier, because convergence of the logarithmic expansion equation (11) does not imply absolute convergence of the path sum equation (16): absolute convergence of equation (16) requires

$$\limsup_l \left[\sum_{[\mathcal{C}_l]} |M_{\mathcal{C}_l}| \right]^{1/l} < \frac{1}{\kappa} \tag{20}$$

whereas absolute convergence of equation (11) requires only

$$\limsup_l |\mathrm{tr} M^l|^{1/l} < \frac{1}{\kappa}, \tag{21}$$

which is a much weaker condition. In fact the latter can easily be estimated in a way independent of the 'volume' N:

$$|\mathrm{tr} M^l|^{1/l} \leqslant N^{1/l} \rho(M)^{1/l} \leqslant N ||M||, \tag{22}$$

but it is in general not true that

$$\sum_{\mathcal{C}_l} |M_{\mathcal{C}_l}| \leqslant N ||M|| \tag{23}$$

and therefore it is difficult to find estimates for $|\mathrm{tr} M^l|^{1/l}|$ independent of N.

So in general there is no simple relation between the zeroes of the factors of equation (19) and those of $\det W$.

In the next section we turn to the main problem of deriving a similar loop formula for the Dirac–Wilson determinant. There the index set is composed of lattice indices and internal ones (spinor, color, flavor) and the definitions of loops etc are modified accordingly. On the other hand, the Dirac–Wilson operator connects only nearest neighbor lattice points, which leads to a tremendous simplification, since instead of the 'flight journeys' considered here, we have only paths consisting of nearest neighbor steps.

3. The loop formula for the Dirac–Wilson determinant

We use here the Wilson fermion formulation in $d = 2$ or 4 dimensions

$$W = \mathbb{1} - \kappa Q = \mathbb{1} - \kappa \sum_{i=1}^{d-1} (\Gamma_{+i} U_i T_i + \Gamma_{-i} T_i^{-1} U_i^{-1}) - \kappa (\mathrm{e}^{\mu} \Gamma_{+4} U_4 T_4 + \mathrm{e}^{-\mu} \Gamma_{-4} T_4^{-1} U_4^{-1})], \tag{24}$$

where κ is the hopping parameter, T_μ are lattice translation, U_μ are link matrices (assumed here in the fundamental representation of $SU(3)$ or $SL(3, C)$), μ is the chemical potential and

$$\Gamma_{\pm\mu} = \mathbb{1} \pm \gamma_\mu,\ \gamma_\mu = \gamma_\mu^*,\ \ \gamma_\mu^2 = \mathbb{1}, \tag{25}$$

$$\mathrm{tr}\Gamma_{\pm\mu} = d,\ \det \Gamma_{\pm\mu} = 0,\ \ \Gamma_{\pm\mu}\Gamma_{\mp\mu} = 0,\ \Gamma_{\pm\mu}^2 = 2\Gamma_{\pm\mu} \tag{26}$$

are Wilson's projectors for the Dirac d.o.f.'s. In this formulation the loop formula takes a simpler form, since back-steps are forbidden. See also appendix A.

The loop formula [11, 12, 15] for a finite simple square ($d = 2$) or hypercubic ($d = 4$) lattice reads:

$$\det W = \det(\mathbb{1} - \kappa Q) = \exp(\mathrm{tr}\ln(\mathbb{1} - \kappa Q)), \tag{27}$$

$$= \exp\left[-\sum_{l=1}^{\infty}\sum_{\{\mathcal{C}_l\}}\sum_{s=1}^{\infty}\frac{g_{\mathcal{C}_l}^s}{s}\,\mathrm{tr}_{D,C}[\mathcal{L}_{\mathcal{C}_l}^s]\right], \tag{28}$$

$$= \prod_{l=1}^{\infty}\prod_{\{\mathcal{C}_l\}}\det_{\mathrm{D,C}}(\mathbb{1} - g_{\mathcal{C}_l}\mathcal{L}_{\mathcal{C}_l}), \tag{29}$$

$$g_{\mathcal{C}_l} = \kappa^l\,(\epsilon\ \mathrm{e}^{N_\tau\mu})^r. \tag{30}$$

Here r is the net winding number of the path in the time direction, with periodic or antiperiodic b.c. and $\epsilon = +1(-1)$ correspondingly. We assume periodic b.c. in the 'spatial' directions. Notice that the terms in Q imply unit steps on the lattice, therefore a lattice path $\mathcal{C}_l$ in equation (29) is a closed chain of lattice points as produced by the lth power of Q in equation (27). Due to the trace condition a path must close, but it can repeatedly visit the same lattice point before closing (the Pauli principle was used in writing the determinant, after that everything is only matrix algebra). Notice that this condition also implies that on an even lattice l is even. In the following we shall speak of the $\mathcal{C}_l$ paths in equations (28) and (29)) as *primary paths*. They are *distinguishable, non-exactly self-repeating lattice closed paths of length l*. Since the primary paths are closed paths and the ensuing $\mathrm{tr}_{D,C}$ do not depend on cyclic permutations of their points we do not identify a starting point for $\mathcal{C}_l$. Nonetheless, in producing such a path we can start at each of its points, therefore we get a multiplicity factor l.

We define the loops $\mathcal{L}_{\mathcal{C}_l}$ as the chains of links and Γ factors on the primary path $\mathcal{C}_l$ and call them *primary loops*. A loop, however, may be repeated in covering the path $\mathcal{C}_l$ before the Dirac and color traces of $\mathrm{tr}_{D,C}$ close, and the exponent s in equation (28) counts the repetitions in covering $\mathcal{C}_l$. Since stepping further after completing the primary path we only obtain identical paths, however, these repetitions do not produce a further multiplicity factor, therefore no further factor s counting the repetitions appears. These considerations have been taken into account in equations (28) and (29).

With λ denoting the links along $\mathcal{C}_l$ we have:

$$\mathcal{L}_{\mathcal{C}_l} = \prod_{\lambda\in\mathcal{C}_l}\Gamma_\lambda U_\lambda = \Gamma_{\mathcal{C}_l}\,U_{\mathcal{C}_l},\quad \Gamma_{\mathcal{C}_l} = \prod_{\lambda\in\mathcal{C}_l}\Gamma_\lambda,\quad U_{\mathcal{C}_l} = \prod_{\lambda\in\mathcal{C}_l}U_\lambda. \tag{31}$$

The Dirac factors $\mathrm{tr}_D\Gamma_{\mathcal{C}_l}$ can be calculated for each $\mathcal{C}_l$ geometrically [15] or numerically.

The contribution of one- and two-dimensional loops $\mathcal{L}_{\tilde{\mathcal{C}}_l}$, $\tilde{\mathcal{C}}_l$: linear and planar loops closing on or over the lattice, in (29) further simplifies to

$$\prod_{l=1}^{\infty}\prod_{\{\tilde{\mathcal{C}}_l\}}\det_{\mathrm{C}}(\mathbb{1} - h_{\tilde{\mathcal{C}}_l}\mathcal{L}_{\tilde{\mathcal{C}}_l})^{d/2}, \tag{32}$$

$$= \prod_{l=1}^{\infty} \prod_{\{\tilde{\mathcal{C}}_l\}} (1 - h_{\tilde{\mathcal{C}}_l} \mathrm{tr}_C U_{\tilde{\mathcal{C}}_l} + h^2_{\tilde{\mathcal{C}}_l} \mathrm{tr}_C U^{-1}_{\tilde{\mathcal{C}}_l} - h^3_{\tilde{\mathcal{C}}_l})^{d/2}, \tag{33}$$

$$h_{\mathcal{C}_l} = g_{\mathcal{C}_l} \times \frac{2}{d} \mathrm{tr}\Gamma_{\mathcal{C}_l} = \kappa^l \, (\epsilon \, \mathrm{e}^{N_\tau \mu})^r \times \frac{2}{d} \mathrm{tr}\Gamma_{\mathcal{C}_l}, \, \det \Gamma_{\mathcal{C}_l} = 0. \tag{34}$$

where equation (33) holds for $SL(3, C)$ ($SU(3)$) gauge group.

In deriving equations (32) and (33) we used the fact that the Dirac and Yang–Mills traces in equation (28) factorize and the former concern products of the projection operators equations (25) and (26):

$$\mathrm{tr}_{D,C}[\mathcal{L}^s_{\mathcal{C}_l}] = \mathrm{tr}\,[\Gamma^s_{\mathcal{C}_l}]\,\mathrm{tr}\,[U^s_{\mathcal{C}_l}], \tag{35}$$

$$\frac{2}{d}\,\mathrm{tr}[\Gamma^s_{\mathcal{C}_l}] = \left[\frac{2}{d}\,\mathrm{tr}\Gamma_{\mathcal{C}_l}\right]^s, \tag{36}$$

where the second equation holds for linear and planar loops [15]. In particular, e.g. for straight Polyakov loops and their inverses $\mathcal{L}_{\mathcal{C}_{N_\tau}} = P$, $\mathcal{L}^{-1}_{\mathcal{C}_{N_\tau}} = P^{-1}$ we have

$$h_P = (2\zeta)^{N_\tau}, \, h_{P^{-1}} = (2\tilde{\zeta})^{N_\tau}, \, \zeta = \kappa \mathrm{e}^{\mu}, \, \tilde{\zeta} = \kappa \mathrm{e}^{-\mu}. \tag{37}$$

For loops which explore more than two dimensions of the lattice equation (36) does not hold generally and therefore equation (29) cannot always be rewritten similar to equation (33).

We shall speak of *basic paths* as primary paths without repeated visitation of a point. Notice that the number of basic paths is finite on a finite lattice. The primary paths are obtained by chaining basic paths in arbitrary order and arbitrarily often repeated (only the resulting chain as a whole should not be repeated). Hence even on a finite lattice there are infinitely many primary paths.

Notice therefore that the rhs of the loop formula on any lattice is an infinite product. Since on a finite lattice the determinant is a polynomial in κ of order $N_{\max} = dN_{\mathrm{L}}N_{\mathrm{c}}$ with $d = 2, 4$ the dimension, N_{L} the lattice volume, N_{c} the number of colors the loop formula implies cancellations of the higher orders. This happens algebraically (i.e. without worrying about convergence), but it is justified analytically only if the nested infinite series involved converge absolutely. See the discussion in section 2.

On the other hand arbitrarily truncating the rhs by keeping only a finite number of factors up to some l_0 will make the loop formula into an approximation of order l_0. The zeroes suggested by the rhs are therefore not true zeroes of the determinant but will corroborate to provide the approximation of the latter to the above order.

When we are dealing with finite temperature and $\mu > 0$, with (anti-)periodic boundary conditions in time the coefficients for loops with positive net winding number $r > 0$ in the time direction will have coefficients containing powers of ζ and can therefore be of order 1 or larger:

$$g_{\mathcal{C}_l} = \kappa^l (e^{N_\tau \mu})^r = \kappa^{l_\sigma} \zeta^{r\,N_\tau}, \, l_\sigma = l - r\,N_\tau \geqslant 0, \, \zeta = \kappa \mathrm{e}^{\mu}. \tag{38}$$

It is useful therefore in this case to reorder the loops according to powers κ^{l_σ}, the lowest order (LO) being given by straight Polyakov loops, the higher orders will involve Polyakov loops 'decorated' by spatial excursions. To LO and NLO we can use equations (33), (34) and (37) for these loops, see [11, 12] for details. Notice that it is important to use here the s-resummation which brings the loops into the determinant factors, in order to avoid large numbers in the exponents. Loops without windings have no ζ factors and start at order κ^4.

For completeness the formal derivation is described in detail in appendix B. In appendix C we illustrate on two simple examples the evaluation of the formula and the problem of the zeroes.

4. Conclusions

The loop formula equations (27)–(29) expresses the determinant of the $\mathcal{N} \times \mathcal{N}$ Wilson fermionic matrix, with $\mathcal{N} = N_{\text{Lattice}} \times N_{\text{Dirac}} \times N_{\text{Color}}$, as an infinite product of determinants of a fixed (independent of N_{Lattice}), small dimension $(N_{\text{Dirac}} \times N_{\text{Color}})^2$. The loop formula is based on the loop expansion given in [15], which itself represents a hopping parameter expansion, and its derivation requires careful reordering of the terms and resummation of infinite nested series. This way one obtains an algebraically correct representation, whose use as an approximation requires, however, a convergence analysis.

The infinite product of small determinants on the rhs can be ordered according to the length of the loops or to the order of the hopping parameter. The individual zeroes of the rhs are not zeroes of the lhs but the rhs converges to the big, exact determinant of the lhs in the region below the lowest zeroes. Truncation of the rhs after a given loop length produces approximations of the lhs after expanding the product and retaining the terms up to the corresponding order.

The formula provides a valid series of approximations in its range of absolute convergence; this domain does, however, not include the zeroes of the full determinant nor those of any approximants. It can thus not be used for an approximate determination of the zeroes of the full determinant, but the approximants can still be useful, e.g. in problems of QCD at finite chemical potential (or density). The formula may be interpreted as providing an ensemble of loops; this suggests that higher orders, which are hard to evaluate algebraically, might be instead be produced by stochastic generation of loops as in [6].

Appendix A. Grand canonical partition function for QCD with Wilson fermions at $\mu > 0$

$$\mathcal{Z}(\beta, \kappa, \gamma_G, \gamma_F, \mu) = \int [DU] \mathrm{e}^{-S_G(\beta,\gamma_G,\{U\})} \mathcal{Z}_F(\kappa, \gamma_F, \mu, \{U\}), \tag{39}$$

$$S_G(\beta, \gamma_G, \{U\}) = -\frac{\beta}{N_c} Re \ \mathrm{tr} \left(\frac{1}{\gamma_G} \sum_{j>i=1}^{3} P_{ij} + \gamma_G \sum_i P_{i4} \right), \tag{40}$$

$$\mathcal{Z}_F(\kappa, \gamma_F, \mu, \{U\}) = \det W(\kappa, \gamma_F, \mu, \{U\}), \tag{41}$$

$$\begin{aligned} W = \mathbb{1} - \kappa Q = \mathbb{1} - \kappa \sum_{i=1}^{3} (\Gamma_{+i} U_i T_i + \Gamma_{-i} T_i^{-1} U_i^{-1}) \\ - \kappa \gamma_F (\mathrm{e}^{\mu} \Gamma_{+4} U_4 T_4 + \mathrm{e}^{-\mu} \Gamma_{-4} T_4^{-1} U_4^{-1}) \end{aligned} \tag{42}$$

$$\Gamma_{\pm\mu} = \mathbb{1} \pm \gamma_\mu, \ \gamma_\mu = \gamma_\mu^*, \ \gamma_\mu^2 = \mathbb{1}, \ \mathrm{tr}\Gamma_{\pm\mu} = 4, \ \det \Gamma_{\pm\mu} = 0 \tag{43}$$

$$\kappa = \frac{1}{2(M + 3 + \gamma_F \cosh \mu)} = \frac{1}{2(M_0 + 3 + \gamma_F)}, \tag{44}$$

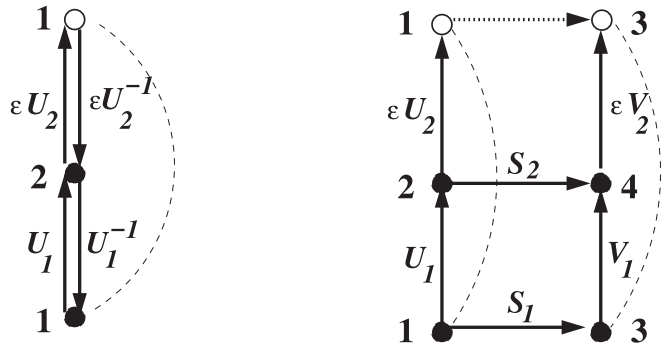

Figure B1. Case 1 and case 2, $\epsilon = 1(-1)$ for periodic boundary conditions (pbc) (apbc), repectively. On the right plot the inverse links are no longer indicated.

(one flavor). Here M is the 'bare mass', M_0 the bare mass at $\mu = 0$, U denote the link variables and T lattice translations. For completeness we introduced anisotropy factors γ_G, γ_F which have to be tuned, by requiring isotropy of physical quantities at $T = \mu = 0$ (hadron masses, string tension etc), defining so a unique physical anysotropy of the lattice spacings $\eta = a_\sigma/a_\tau$ which will enter the physical temperature, chemical potential, masses, etc. In the following $\gamma_G = \gamma_F = 1$. The exponential prescription for μ ensures cancelling of divergences in the small a limit.

Appendix B. Formal derivation

In the following we shall formally derive the loop formula. For illustration we shall refer to figure B1 . With

$$W = \mathbb{1} - \kappa\, Q \tag{45}$$

we have at first step

$$\mathrm{tr}\ln W = \mathrm{tr}\ln(\mathbb{1} - \kappa Q) = -\sum_{l=1}^{\infty}\frac{\kappa^n}{n}\mathrm{tr}(Q^n), \tag{46}$$

$$= -\sum_{l=1}^{\infty}\sum_{\{\mathcal{C}_l\}}\sum_{s=1}^{\infty}\frac{g^s_{\mathcal{C}_l}}{s}\,\mathrm{tr}_{D,C}[\mathcal{L}^s_{\mathcal{C}_l}], \tag{47}$$

Notice:

(1) Due to the traces we obtain only connected, closed loops on the lattice (without backtracking for Wilson fermions).

(2) We consider now *primary loops* $\mathcal{L}_{\mathcal{C}_l}$. They are of all possible lengths l and may close on or over the toroidal lattice.

Example. In $\frac{1}{4}\mathrm{tr}Q^4$ the loop $\mathrm{tr}_{D,C}S_1V_1S_2^{-1}U_2$ appears with multiplicity 4 (since it can be started at each node) while leading each time to the same contribution by cyclicity of $\mathrm{tr}_{D,C}$— see figure B1 right.

(3) Each primary loop can appear repeated any number of times s. As already pointed out, because of the indistiguishability in the repetition there is no further multiplicity.

Example. On a 1-dim periodic lattice of length 2 the term $\frac{1}{2}\mathrm{tr}Q^2$ has 2 loops closing over the lattice, U_1U_2 and U_2U_1, contributing equally $\mathrm{tr}_{D,C}U_1U_2 = \mathrm{tr}_{D,C}U_2U_1$, see figure B1 left. Since the corresponding paths are identical we speak of them as one primary loop with multiplicity 2, contributing thus $2 \times \frac{1}{2}\mathrm{tr}_{D,C}U_1U_2$ in the loop formula.

The term $\frac{1}{4}\mathrm{tr}Q^4$ of order $n = 4$ contains $(U_1 + U_2)^4$ and has just 2 different terms, $U_1U_2U_1U_2$ and $U_2U_1U_2U_1$, allowed by the lattice trace (notice that a term like $U_1^2U_2^2$ cannot appear since each link is associated with a lattice step). One can see $U_2U_1U_2U_1$ as the first permutation of $U_1U_2U_1U_2$, further permutations do not bring new loops. The contributions of these two loops to the ensuing $\mathrm{tr}_{D,C}$ are equal by cyclicity of the trace. Therefore we speak here of just one loop, say $U_1U_2U_1U_2$. Since it is a genuine repetition it does not represent a primary loop, but the $s = 2$ repetition $(U_1U_2)^2$ of the primary loop U_1U_2 of multiplicity 2 and contributes therefore $2 \times \frac{1}{4}\mathrm{tr}_{D,C}[(U_1U_2)^2]$ in the loop sum.

(4) Identification of primary loops.

Example. The loops $S_1V_1S_2^{-1}U_1^{-1}$ and $S_1V_1S_2^{-1}U_2$ are basic loops, the first one is a plaquette, the second a decorated Polyakov loop, both appearing with multiplicity 4. $S_1V_1S_2^{-1}U_1^{-1}S_1V_1S_2^{-1}U_1^{-1}S_1V_1S_2^{-1}U_2$ is a primary loop, consisting of the iterated basic loop Plaquette and the basic Polyakov loop and has multiplicity 12. In contrast to this, the loop $S_1V_1S_2^{-1}U_1^{-1}S_1V_1S_2^{-1}U_1^{-1}S_1V_1S_2^{-1}U_1^{-1}$ is not a primary loop but the $s = 3$ repetion of the basic loop plaquette $S_1V_1S_2^{-1}U_1^{-1}$ and appears therefore with multiplicity 4.

We now make the 2nd step to obtain:

$$\det W = \exp(\mathrm{tr}\ln W), \tag{48}$$

$$= \prod_{l=1}^{\infty}\prod_{\{\mathcal{C}_l\}}\det_{\mathrm{D,C}}(\mathbb{1} - g_{\mathcal{C}_l}\mathcal{L}_{\mathcal{C}_l}). \tag{49}$$

Notice:

(5) The summands in equation (47) are Dirac and color traces, that is just complex numbers. Each summand corresponding to a primary loop is of the form

$$-\sum_{s=1}^{\infty}\frac{g_{\mathcal{C}_l}^s}{s}\mathrm{tr}_{D,C}[\mathcal{L}_{\mathcal{C}_l}^s] = \mathrm{tr}_{D,C}\ln(\mathbb{1} - g_{\mathcal{C}_l}\mathcal{L}_{\mathcal{C}_l}) \tag{50}$$

since this is just the expansion of a logarithm (*s-resummation*). It is the repeated covering of primary loops which resumes to the logarithms.

(6) Using equation (50) we can rewrite equation (47) as

$$-\sum_{l=1}^{\infty}\sum_{\{\mathcal{C}_l\}}\mathrm{tr}_{D,C}\ln(\mathbb{1} - g_{\mathcal{C}_l}\mathcal{L}_{\mathcal{C}_l}) \tag{51}$$

Since the summands commute we can exponentiate and rewrite equation (47) as

$$\prod_{l=1}^{\infty}\prod_{\{\mathcal{C}_l\}}\mathrm{e}^{\mathrm{tr}_{D,C}\ln(\mathbb{1}-g_{\mathcal{C}_l}\mathcal{L}_{\mathcal{C}_l})} = \prod_{l=1}^{\infty}\prod_{\{\mathcal{C}_l\}}\det_{\mathrm{D,C}}(\mathbb{1} - g_{\mathcal{C}_l}\mathcal{L}_{\mathcal{C}_l}), \tag{52}$$

by inverting the general formula equation (50). This is equation (29).

As remarked before, the zeroes suggested by the rhs factors of the loop formula generally are not zeroes of the lhs, but keeping the LO factors up to some order may provide reasonable approximations for the lhs, depending on the parameters and configuration.

A particular situation occurs for the case of large mass and chemical potential. Here the temporal loops disentangle in the limit $\kappa \to 0$, $\mu \to \infty$, $\zeta = \kappa e^{\mu} = fixed$ and the determinant reduces to the 'HD-QCD' determinant in LO, here for antiperiodic boundary conditions (apbc) ($\epsilon = -1$)

$$D_{HD}^{[0]} = \prod_{\vec{x}} \det_C (1 + CP_{\vec{x}})^{d/2} \det_C (1 + \tilde{C}P_{\vec{x}}^{-1})^{d/2}\,, \tag{53}$$

$$= (1 + C\mathrm{tr}P_{\vec{x}} + C^2\mathrm{tr}P_{\vec{x}}^{-1} + C^3)^{d/2} \times (1 + \tilde{C}\mathrm{tr}P_{\vec{x}}^{-1} + \tilde{C}^2\mathrm{tr}P_{\vec{x}} + \tilde{C}^3)^{d/2}, \tag{54}$$

$$g_P \equiv C = (2\kappa e^{\mu})^{N_t}, \quad g_{P^{-1}} \equiv \tilde{C} = (2\kappa e^{-\mu})^{N_t}. \tag{55}$$

Notice that since in this case the expansion parameter C can be of order 1, the s-resummation to equation (29) is important for avoiding large exponents in equation (28).

Appendix C. Simple examples

In the following we treat two examples which both illustrate the loop formula and its problems.

We use

$$\Gamma_{\pm,1} = \mathbb{1} \pm \sigma_1 = \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}, \quad \Gamma_{\pm,2} = \mathbb{1} \pm \sigma_2 = \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}, \tag{56}$$

$$\Gamma_{\pm,i}^2 = 2\,\Gamma_{\pm,i}, \quad \Gamma_{\pm,i}\Gamma_{\mp,i} = 0, \quad \mathrm{tr}\Gamma_{\pm,i} = 2, \quad \det\Gamma_{\pm,i} = 0 \tag{57}$$

$$\Gamma_{\epsilon_1,1}\,\Gamma_{\epsilon_2,2} = -\Gamma_{\epsilon_2,2}\,\Gamma_{\epsilon_1,1} + 2\Gamma_{\epsilon_1,1} + 2\Gamma_{\epsilon_2,2} - 2\,\mathbb{1}, \tag{58}$$

$$= \Gamma_{\epsilon_2,2}\,\Gamma_{-\epsilon_1,1} + 2\,\Gamma_{-\epsilon_1,1} - 2\,\mathbb{1}, \tag{59}$$

$$\zeta = \kappa e^{\mu}, \quad \tilde{\zeta} = \kappa e^{-\mu}, \quad \zeta\,\tilde{\zeta} = \kappa^2, \tag{60}$$

mimicking a 2-dim lattice theory at finite μ. We use U(1) links to reduce the dimensionality of W.

Notice that we have

$$\mathcal{L}_{\mathcal{C}_l} = \Gamma_{\mathcal{C}_l} \prod_{\lambda\in\mathcal{C}_l} U_\lambda,\ \Gamma_{\mathcal{C}_l} = \prod_{\lambda\in\mathcal{C}_l} \Gamma_\lambda,\ \Gamma_\lambda = \Gamma_{\pm,i},\ \det\Gamma_{\mathcal{C}_l} = 0, \tag{61}$$

$$\det_{\mathrm{D,C}} (\mathbb{1} - g_{\mathcal{C}_l}\mathcal{L}_{\mathcal{C}_l}) = 1 - h_{\mathcal{C}_l}\mathcal{L}_{\mathcal{C}_l}, \tag{62}$$

see also equation (34). Since we are interested in chemical potential problems we keep ζ fixed in this illustrations and consider various κ orders.

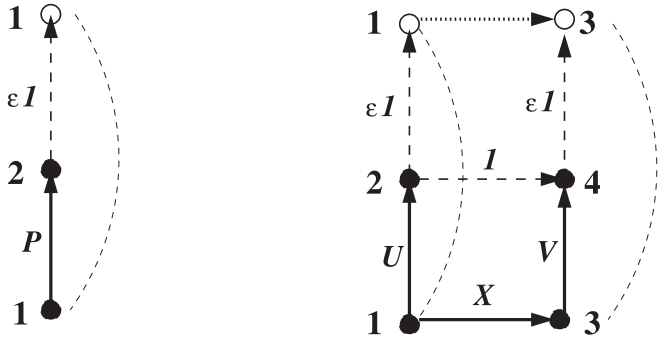

Figure C1. Case 1 and case 2, maximal gauge fixing, $\epsilon = 1(-1)$ for pbc (apbc). Inverse links are not indicated on the plots.

C.1. Case (1)

We consider a Polyakov loop $P = U_1\, U_2$

$$W = \begin{pmatrix} 1 & 0 & \zeta\, U_1 + \epsilon\tilde{\zeta}\, U_2^{-1} & \zeta\, U_1 - \epsilon\tilde{\zeta}\, U_2^{-1} \\ 0 & 1 & \zeta\, U_1 - \epsilon\tilde{\zeta}\, U_2^{-1} & \zeta\, U_1 + \epsilon\tilde{\zeta}\, U_2^{-1} \\ \epsilon\zeta\, U_2 + \tilde{\zeta}\, U_1^{-1} & \epsilon\zeta\, U_2 - \tilde{\zeta}\, U_1^{-1} & 1 & 0 \\ \epsilon\zeta\, U_2 - \tilde{\zeta}\, U_1^{-1} & \epsilon\zeta\, U_2 + \tilde{\zeta}\, U_1^{-1} & 0 & 1 \end{pmatrix}. \tag{63}$$

Since the determinant is gauge invariant we use gauge transformations to put the Polyakov loop on one link, figure C1 left plot:

$$W = \begin{pmatrix} 1 & 0 & \zeta\, P + \epsilon\tilde{\zeta}\, 1 & \zeta\, P - \epsilon\tilde{\zeta}\, 1 \\ 0 & 1 & \zeta\, P - \epsilon\tilde{\zeta}\, 1 & \zeta\, P + \epsilon\tilde{\zeta}\, 1 \\ \epsilon\zeta\, 1 + \tilde{\zeta}\, P^{-1} & \epsilon\zeta\, 1 - \tilde{\zeta}\, P^{-1} & 1 & 0 \\ \epsilon\zeta\, 1 - \tilde{\zeta}\, P^{-1} & \epsilon\zeta\, 1 + \tilde{\zeta}\, P^{-1} & 0 & 1 \end{pmatrix}. \tag{64}$$

E.g., with $\epsilon = -1$ for apbc simple algebra gives

$$\begin{aligned} \det W &= 1 + 4\zeta^{\,2}P + 4\tilde{\zeta}^{\,2}P^{-1} + 16\kappa^4 = 1 + 4\zeta^{\,2}P + O(\kappa^4), \\ P &= \mathrm{tr}U_1U_2, \quad P^{-1} = \mathrm{tr}U_2^{-1}U_1^{-1} \end{aligned} \tag{65}$$

It can have real zeroes if P, P^{-1} have large, negative real parts. The primary loops are $P = U_1U_2$ and $P^{-1} = U_2^{-1}U_1^{-1}$ and the loop formula equation (29) *with* s-resumation gives

$$\begin{aligned} \det W &= \mathrm{e}^{\ln(1+4\zeta^{\,2}P)+\ln(1+4\tilde{\zeta}^{\,2}P^{-1})} = (1 + 4\zeta^{\,2}P)(1 + 4\tilde{\zeta}^{\,2}P^{-1}) \\ &= 1 + 4\zeta^{\,2}P + 4\tilde{\zeta}^{\,2}P^{-1} + 16\kappa^4 \end{aligned} \tag{66}$$

which coincides with the exact result equation (65). Notice that without s-resummation the higher orders do not cancel and we would have obtained wrong results already at 0th order in κ:

$$\begin{aligned} s = 1 : \mathrm{e}^{4\zeta^{\,2}P+4\tilde{\zeta}^{\,2}P^{-1}} &= 1 + 4\zeta^{\,2}P + 8\zeta^{\,4}P^2 \\ &+ 4\tilde{\zeta}^{\,2}P^{-1} + 16\kappa^4 + 8\tilde{\zeta}^{\,4}(P^{-1})^2 + \cdots, \end{aligned} \tag{67}$$

$$s = 1 + 2 : \mathrm{e}^{4\zeta\ {}^2P - 8\zeta\ {}^4P^2 + 4\tilde{\zeta}\ {}^2P^{-1} - 8\tilde{\zeta}\ {}^4(P^{-1})^2}$$
$$= 1 + 4\zeta\ {}^2P - \frac{80}{3}\zeta\ {}^6P^3 + 4\tilde{\zeta}\ {}^2P^{-1} + 16\kappa^4 - \frac{80}{3}\tilde{\zeta}\ {}^6(P^{-1})^3 + \cdots. \tag{68}$$

C.2. Case (2)

We consider two short, connected Polyakov loops (a 2×2 lattice with pbc in 1 direction and free b.c in the other). See figure B1, right plot and table C1 . Using again gauge transformations to a maximal gauge fixing we obtain figure C1, right plot and table C2 . We have $\epsilon = 1(-1)$ for pbc(apbc).

Basic and primary loops of length up to $l = 6$, with corresponding Dirac coefficients C_i and multiplicities m_i:

$$\begin{aligned} &l = 2 : \\ &1 - 2 - 1 : U_1 U_2 = L_1,\ C_1 = 4\epsilon\zeta\ {}^2,\ \ m_1 = 2 \\ &3 - 4 - 3 : V_1 V_2 = L_2, C_2 = 4\epsilon\zeta\ {}^2, m_2 = 2, \end{aligned} \tag{69}$$

$$\begin{aligned} &l = 4: \\ &1 - 3 - 4 - 2 - 1 : S_1 V_1 S_2^{-1} U_1^{-1} = L_3, \quad C_3 = -4\kappa^4,\ \ m_3 = 4 \\ &2 - 4 - 3 - 1 - 2 : S_2 V_2 S_1^{-1} U_2^{-1} = L_4, \quad C_4 = -4\kappa^4,\ \ m_4 = 4 \\ &1 - 3 - 4 - 2 - 1 : \quad S_1 V_1 S_2^{-1} U_2 = L_5, C_5 = 4\epsilon\zeta\ {}^2\kappa^2,\ \ m_5 = 4 \\ &2 - 4 - 3 - 1 - 2 : \quad S_2 V_2 S_1^{-1} U_1 = L_6, C_6 = 4\epsilon\zeta\ {}^2\kappa^2, m_6 = 4, \end{aligned} \tag{70}$$

$$\begin{aligned} &l = 6 : \\ &1 - 3 - 4 - 2 - 1 - 2 - 1 : \ S_1 V_1 S_2^{-1} U_2 U_1 U_2 = L_7,\ C_7 = -16\zeta\ {}^4\kappa^2,\ \ m_7 = 6 \\ &2 - 4 - 3 - 4 - 2 - 1 - 2 : S_2 V_2 S_1^{-1} U_1 U_2 U_1 = L_8,\ C_8 = -16\zeta\ {}^4\kappa^2,\ \ m_8 = 6 \\ &1 - 3 - 4 - 3 - 4 - 2 - 1 : \ S_1 V_1 V_2 V_1 S_2^{-1} U_2 = L_9,\ C_9 = -16\zeta\ {}^4\kappa^2,\ \ m_9 = 6 \\ &2 - 4 - 3 - 4 - 3 - 1 - 2 : S_2 V_2 V_1 V_2 S_1^{-1} U_1 = L_{10}, C_{10} = -16\zeta\ {}^4\kappa^2, m_{10} = 6 \\ &1 - 3 - 4 - 3 - 1 - 2 - 1 : S_1 V_1 V_2 S_1^{-1} U_1 U_2 = L_{11}, C_{11} = -16\zeta\ {}^4\kappa^2, m_{11} = 6 \\ &2 - 4 - 3 - 4 - 2 - 1 - 2 : S_2 V_2 V_1 S_2^{-1} U_2 U_1 = L_{12}, C_{12} = -16\zeta\ {}^4\kappa^2, m_{12} = 6 \end{aligned} \tag{71}$$

(and their inverses[4]).

The loop formula gives:

$$\det W = \prod_{i=1}^{\infty} (1 - C_i L_i)\,(1 - C_i L_i^{-1}), \tag{72}$$

where L_i are primary loops, obtained by chaining together basic loops.

For calculations we can use the maximal gauge, setting $U_2 = V_2 = S_2 = 1$, see table C2. Then $L_1 = U$, $L_2 = V$ are the two straight Polyakov loops, XVU^{-1} and X the two plaquettes L_3, L_4 and $L_5 = V\ X$, $L_6 = U\ X^{-1}$ are *basic* decorated Polyakov loops and $L_7 = X\ V\ U$, $L_8 = X^{-1}U^2$, $L_9 = X\ V^2$, $L_10 = X^{-1}U\ V$, $L_{11} = L_{12} = U\ V$ are *primary* decorated Polyakov loops obtained by chaining a basic straight Polyakov loop and a basic decorated Polyakov loop. Notice that we can attach further straight Polyakov loops and obtain primary loops of arbitrary length in order κ^2 but we stopped at length $l = 6$ for this illustration.

[4] We thank Dr Elmar Bittner for writing the program to produce basic and primary loops.

Table C1. Fermionic matrix W from figure B1, right plot.

$$\begin{pmatrix}
1 & 0 & \zeta U_1+\tilde{\zeta}\,\epsilon U_2^{-1} & \zeta U_1-\tilde{\zeta}\,\epsilon U_2^{-1} & \kappa S_1 & -\mathrm{i}\kappa S_1 & 0 & 0\\
0 & 1 & \zeta U_1-\tilde{\zeta}\,\epsilon U_2^{-1} & \zeta U_1+\tilde{\zeta}\,\epsilon U_2^{-1} & \mathrm{i}\kappa S_1 & \kappa S_1 & 0 & 0\\
\zeta\,\epsilon U_2+\tilde{\zeta}\,U_1^{-1} & \zeta\,\epsilon U_2-\tilde{\zeta}\,U_1^{-1} & 1 & 0 & 0 & 0 & \kappa S_2 & -\mathrm{i}\kappa S_2\\
\zeta\,\epsilon U_2-\tilde{\zeta}\,U_1^{-1} & \zeta\,\epsilon U_2+\tilde{\zeta}\,U_1^{-1} & 0 & 1 & 0 & 0 & \mathrm{i}\kappa S_2 & \kappa S_2\\
\kappa S_1^{-1} & \mathrm{i}\kappa S_1^{-1} & 0 & 0 & 1 & 0 & \zeta V_1+\tilde{\zeta}\,\epsilon V_2^{-1} & \zeta V_1-\tilde{\zeta}\,\epsilon V_2^{-1}\\
-\mathrm{i}\kappa S_1^{-1} & \kappa S_1^{-1} & 0 & 0 & 0 & 1 & \zeta V_1-\tilde{\zeta}\,\epsilon V_2^{-1} & \zeta V_1+\tilde{\zeta}\,\epsilon V_2^{-1}\\
0 & 0 & \kappa S_2^{-1} & \mathrm{i}\kappa S_2^{-1} & \zeta\,\epsilon V_2+\tilde{\zeta}\,V_1^{-1} & \zeta\,\epsilon V_2-\tilde{\zeta}\,V_1^{-1} & 1 & 0\\
0 & 0 & -\mathrm{i}\kappa S_2^{-1} & \kappa S_2^{-1} & \zeta\,\epsilon V_2-\tilde{\zeta}\,V_1^{-1} & \zeta\,\epsilon V_2+\tilde{\zeta}\,V_1^{-1} & 0 & 1
\end{pmatrix}$$

Table C2. Maximal gauge $W - 1 = -\kappa\, Q$, see figure C1, right plot.

$$\begin{pmatrix}
0 & 0 & \zeta\, U+\tilde{\zeta}\,\epsilon & \zeta\, U-\tilde{\zeta}\,\epsilon & \kappa X & -\mathrm{i}\kappa X & 0 & 0\\
0 & 0 & \zeta\, U-\tilde{\zeta}\,\epsilon & \zeta\, U+\tilde{\zeta}\,\epsilon & \mathrm{i}\kappa X & \kappa X & 0 & 0\\
\zeta\,\epsilon+\tilde{\zeta}\,U^{-1} & \zeta\,\epsilon-\tilde{\zeta}\,U^{-1} & 0 & 0 & 0 & 0 & \kappa & -\mathrm{i}\kappa\\
\zeta\,\epsilon-\tilde{\zeta}\,U^{-1} & \zeta\,\epsilon+\tilde{\zeta}\,U^{-1} & 0 & 0 & 0 & 0 & \mathrm{i}\kappa & \kappa\\
\kappa X^{-1} & \mathrm{i}\kappa X^{-1} & 0 & 0 & 0 & 0 & \zeta\, V+\tilde{\zeta}\,\epsilon & \zeta\, V-\tilde{\zeta}\,\epsilon\\
-\mathrm{i}\kappa X^{-1} & \kappa X^{-1} & 0 & 0 & 0 & 0 & \zeta\, V-\tilde{\zeta}\,\epsilon & \zeta\, V+\tilde{\zeta}\,\epsilon\\
0 & 0 & \kappa & \mathrm{i}\kappa & \zeta\,\epsilon+\tilde{\zeta}\,V^{-1} & \zeta\,\epsilon-\tilde{\zeta}\,V^{-1} & 0 & 0\\
0 & 0 & -\mathrm{i}\kappa & \kappa & \zeta\,\epsilon-\tilde{\zeta}\,V^{-1} & \zeta\,\epsilon+\tilde{\zeta}\,V^{-1} & 0 & 0
\end{pmatrix}$$

Since the lattice is even we take only even power of Q in evaluating equation (72). We shall only consider loops up to length 6. To order 0 in κ we find:

$$\frac{1}{2n}Q^{2n}=\frac{1}{n}(4\zeta\ \epsilon U)^n+\frac{1}{n}(4\tilde{\zeta}\ \epsilon U^{-1})^n+\frac{1}{n}(4\zeta\ \epsilon V)^n+\frac{1}{n}(4\tilde{\zeta}\ \epsilon V^{-1})^n. \quad (73)$$

Notice that these basic loops are of lengths 2 , $U=U_1U_2$, $V=V_1V_2$. The calculation was performed up to order 16 , which is $2n_{\max}$ and one can check that up to this order there is no deviation from the logarithm series. Assuming that the series continues as expected we find to this order

$$\begin{aligned}D^{[0]}&=(1-4\epsilon\zeta^{\ 2}U)(1-4\epsilon\zeta^{\ 2}V)(1-4\epsilon\tilde{\zeta}^{\ 2}U^{-1})(1-4\epsilon\tilde{\zeta}^{\ 2}V^{-1})\\&=1-4\epsilon\zeta^{\ 2}(U+V)+16\zeta^{\ 4}U\ V+O(\kappa^4).\end{aligned} \quad (74)$$

A similar result follows from the decorated Polyakov loops which appear with a factor κ^2 (for simplicity we no longer write the non-dominant inverse loops which have factors $\tilde{\zeta}$):

$$\begin{aligned}D_1^{[2]}&=(1-4\epsilon\zeta^{\ 2}\kappa^2U\ X^{-1})(1-4\epsilon\zeta^{\ 2}\kappa^2V\ X)\\&=1-4\epsilon\kappa^2\zeta^{\ 2}(V\ X+U\ X^{-1})+O(\kappa^4).\end{aligned} \quad (75)$$

Finaly the non-basic primary loops L_7-L_{12} give

$$\begin{aligned}D_2^{[2]}&=(1-16\zeta^{\ 4}\kappa^2X\ V\ U)(1-16\zeta^{\ 4}\kappa^2X^{-1}\ U^2)\\&\quad\times(1-16\zeta^{\ 4}\kappa^2X\ V^2)(1-16\zeta^{\ 4}\kappa^2X^{-1}U\ V)\\&\quad\times(1-16\zeta^{\ 4}\kappa^2U\ V)^2\\&=1-16\kappa^2\zeta^{\ 4}(2\ U\ V+X\ V\ U+X^{-1}U\ V+X^{-1}U^2+X\ V^2))+O(\kappa^4).\end{aligned} \quad (76)$$

We obtain thus the HD determinant to order κ^2 including all loops up to length 6, in complete agreement with the exact determinant to this order

$$\begin{aligned} D^{[2]} &= (1 - 4\epsilon\zeta^{\,2}(U + V) + 16\zeta^{\,4}U\,V)(1 - 4\epsilon\zeta^{\,2}\kappa^2(X\,V + X^{-1}U)) \\ &\quad \times (1 - 16\zeta^{\,4}\kappa^2(U\,V(2 + X + X^{-1}) + X^{-1}U^2 + X\,V^2)) \\ &= 1 - 4\epsilon\zeta^{\,2}(U + V) + 16\zeta^{\,4}U\,V \\ &\quad - 4\epsilon\zeta^{\,2}\kappa^2(V\,X + U\,X^{-1}) - 32\zeta^{\,4}\kappa^2 U\,V, \end{aligned} \tag{77}$$

$$\begin{aligned} D^{[2]}_{\text{exact}} &= 1 - 4\epsilon\zeta^{\,2}(U + V) + 16\zeta^{\,4}U\,V \\ &\quad - 4\epsilon\zeta^{\,2}\kappa^2(V\,X + U\,X^{-1}) - 32\zeta^{\,4}\kappa^2 U\,V. \end{aligned} \tag{78}$$

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