

PAPER • OPEN ACCESS

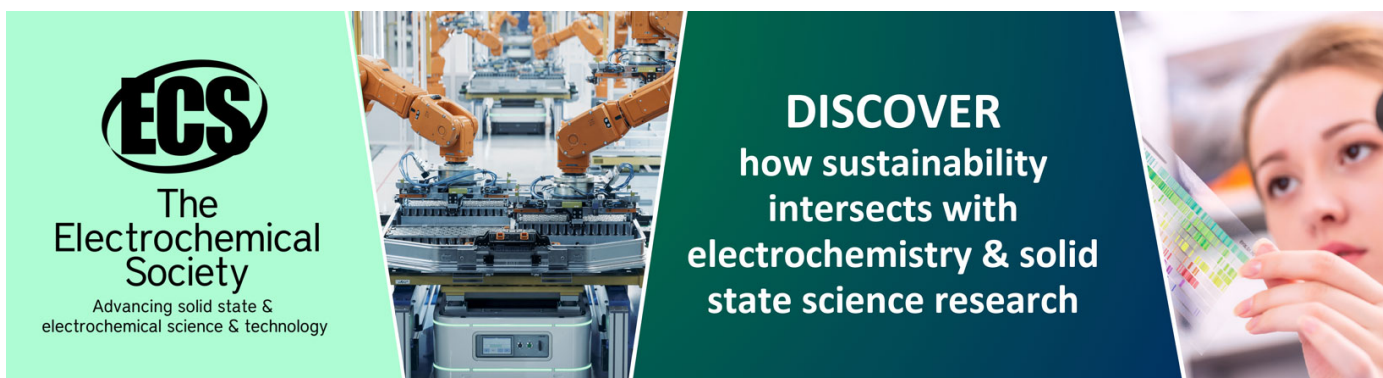
On the conditioning of two-equation road traffic models

To cite this article: M Maciejewski 2018 *IOP Conf. Ser.: Mater. Sci. Eng.* **421** 022021

View the [article online](#) for updates and enhancements.

You may also like

- [Proposals for the reorganization of road traffic in the central area of Pitesti municipality based on microsimulation-performed traffic analyses](#)
A A Boroiu, E Neagu, A Boroiu et al.
- [A new cellular automaton for signal controlled traffic flow based on driving behaviors](#)
Yang Wang, , Yan-Yan Chen et al.
- [Traffic flow velocity disturbance characteristics and control strategy at the bottleneck of expressway](#)
Jun-Wei Zeng, , Yong-Sheng Qian et al.



ECS
The
Electrochemical
Society
Advancing solid state &
electrochemical science & technology

DISCOVER
how sustainability
intersects with
electrochemistry & solid
state science research

On the conditioning of two-equation road traffic models

M Maciejewski

Wydział Maszyn Roboczych i Transportu, Politechnika Poznańska,
ul. Piotrowo 3, 60-965 Poznań, Poland

E-mail: marek.maciejewski@put.poznan.pl

Abstract. This article deals with two-equation macroscopic traffic models, which in addition to the standard continuity equation (the LWR model), contain an additional differential equation expressing the traffic dynamics, specifically the process of speed adaptation to road traffic conditions. There are many two-equation models with various forms of the traffic dynamics equation with different possibilities to achieve an accurate, stable and effective solution. The most popular models have been collected here and presented in a unified matrix form. The quality of particular two-equation traffic models, and thus the quality of the results obtained, depends on the quality of the model itself, and on the discretization and approximation of the problem. In this article, the evaluation of particular models is considered only on a continuous level, i.e. at the level of theoretical description of the models. Standard conditioning measures such as the spectral radius and the spectral condition number were used for this purpose. The results of the estimations have been used to compare particular models as well as to characterize two basic classes of models, i.e. isotropic (symmetrical) and anisotropic (asymmetric) models, and to evaluate the stability and potential computational efficiency of solutions of these two-equation models.

1. Introduction

Road traffic models present phenomena and processes of road traffic in the form of mathematical relations. These models are usually classified according to the degree of idealization (or level of abstraction) of the traffic description, while distinguishing two basic forms, i.e. macroscopic and microscopic models. Microscopic (individualized) models are based on the interaction of vehicles (drivers) with each other, as well as with the surrounding road infrastructure, while macroscopic (aggregated) models do not refer directly to this type of dependency, but describe traffic as a flow of a certain (fictional) continuous medium, presented in phenomenological approach, i.e. without delving into its discrete (microscopic) structure. In fluid mechanics, this invisible discrete internal structure is created by molecules, and respectively, by vehicles in road traffic flow. Further considerations in this article refer only to macroscopic models.

In macroscopic models, the flow of continuous medium is characterised by some specific properties, which is usually described in the (differential or integral) calculus, using aggregated quantities (variables) such as intensity, density and velocity for this purpose. Due to the continuous model of that medium, a direct solution to the compressible flow problem is possible only on the basis of the use of mathematical analysis methods. However, this is not a real approach when considering more complex road systems. Then the only remaining solution is the discretization of the problem (describing its behaviour not in relation to all, infinitely many points of the area, but only to selected points of the system defined usually by the computational mesh) and its approximation (expression of the differential



operators through the corresponding operations on functions), which in the result leads to the solution of a system of algebraic equations (approximating the initial system of differential equations). Such approximate solutions are usually achieved using numerical methods which are characterised by high computational efficiency (in comparison to analytical methods).

The quality of the macroscopic traffic models is determined primarily by the adequacy of the achieved results, while the quality of the simulation results depends on the accuracy and stability of the solution methods and numerical algorithms that were used. This also means that the solution quality is determined, of the one part, by the formulations of the model itself, and, of the second part, by the discretization and approximation of the problem. In this publication, we will refer only to the first of both issues, and thus to the potential ability of the system of model equations to be robust and effective.

On the macroscopic level, many different models are used to describe the traffic flow, characterised by various equation forms with very different possibilities to achieve an accurate, stable and effective solution. Obtaining the above mentioned solution features at a satisfactory level depends among others on an assessment of model formulations on the theoretical (continuous) level. For such an evaluation, it is justified to use standard conditioning measures such as the spectral radius and the spectral condition number.

In the first order, the article collects and presents in a unified form the basic formulations of various traffic flow models. On this basis, all matrix coefficients for different traffic flow models are defined. Determining the eigenvalues for individual classes and subclasses of models is the basis to define the matrix conditioning measures: spectral radiuses and spectral condition numbers. These measures are then used to carry out model comparisons, and thus to evaluate the potential computational efficiency of macroscopic road traffic models.

2. Overview of macroscopic models

The objective of macroscopic models is to describe as closely as possible all phenomena related to the road traffic flow by expressing them through aggregated quantities (i.e. flow rate, density and speed) characterizing the traffic flow in isolation from the behaviour of individual vehicles. The process of creating and improving traffic models was a long-term process which gradually led to more and more perfect mathematical descriptions of traffic flow. The first models referred only to simple traffic situations and failed in the case of more complex traffic conditions and road configurations. They were called first-order, LWR or single-equation models [1, 2]. The single-equation models proved defective because it expressed the local equilibrium conditions in a static manner, i.e. a change in boundary conditions immediately resulted in a new equilibrium state, without any evolution between the previous and new traffic state, i.e. without reflecting the flow dynamics.

In order to correct LWR models, two-equation (second-order) models have been developed that contain an additional differential equation (equivalent to the momentum equation from fluid dynamics) expressing the traffic dynamics, i.e. describing the speed adaptation to current traffic conditions. The form of two-equation models has a significant similarity to the description of compressible fluids (gases). However, this is not a complete similarity due to:

- anticipating reaction of drivers in relation to traffic conditions down the road,
- asymmetry (anisotropy) of the drivers' reactions in relation to the traffic condition down and up the road.

Described drivers' behaviour modifies standard convective interactions, and also takes into account diffusive interactions referred to as traffic relaxation, i.e. toning down the tendency to quickly achieve an equilibrium state.

Because traffic model formulations are mainly dependent on convective effects, in further considerations on the quality and efficiency of these models, diffusive effects will be omitted, which will sometimes result in certain changes in the model descriptions relative to their original formulations. Basic traffic models written in such form will be briefly presented in the following paragraphs as specific cases of a generalised and unified traffic description.

The baseline traffic flow model was adopted in the following form:

$$\begin{bmatrix} \partial_t k \\ \partial_t u \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \partial_x k \\ \partial_x u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (1)$$

where individual symbols mean: t – the time, x – the space (the road), k – the density, u – the speed, $\partial_\alpha(\cdot) = \frac{\partial(\cdot)}{\partial\alpha}$ and $(\cdot)_\beta = \frac{d(\cdot)}{d\beta}$ (where α is t or x , and β – k or u) – (partial and ordinary) differential operators. In turn, the quantities a_{11} , a_{12} , a_{21} and a_{22} are certain generalised (at this moment, still undefined) traffic parameters of models.

Two-equation traffic models consist of the flow continuity equation (the LWR model) as a permanent element of this system, and the traffic dynamics equation that determines the final form and properties of the system. The LWR model has the following standard form:

$$\partial_t k + u \partial_x k + k \partial_x u = 0, \quad (2)$$

which causes that in the system of equations (1) $a_{11} = u$, and $a_{12} = k$, and thus it takes the form

$$\begin{bmatrix} \partial_t k \\ \partial_t u \end{bmatrix} + \begin{bmatrix} u & k \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \partial_x k \\ \partial_x u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (3)$$

The above, general form of the two-equation macroscopic model is the basis for description and analysis of many existing solutions in this field. The individual models will be defined below depending on the form of quantities a_{21} and a_{22} .

Similarly to the continuity equation, the road traffic dynamics equation refers quite clearly to the momentum conservation equation, and therefore in its formulation, the quantities and terms characteristic for the Navier-Stokes equation occur. Because the diffusive effects (in this publication) have been omitted by default, the traffic dynamics equation shows a significant similarity to the Euler equation for compressible fluids. For this reason, the concept of pressure p was introduced to the traffic dynamics description. This pressure characterises the anticipation, and the speed of sound c_0 is understood as the propagation speed of traffic disturbances. The pressure p (as a predictive factor) depends on the traffic density (and possibly speed), which may have appropriate consequences for the speed of sound c_0 . Formally (according to the fluid dynamics), the speed of sound is defined as

$$c_0 = \sqrt{\frac{dp}{dk}} \equiv \sqrt{p_k} \quad (4)$$

and is one of the important distinguishing features of traffic models, which will be briefly presented below.

The PW (Payne–Whitham) model [3, 4], in the version without diffusive terms, is characterised by the following forms of a_{21} and a_{22} (3):

$$a_{21} = -\frac{1}{2\tau} u_k \quad a_{22} = u, \quad (5)$$

where τ is the time constant. In the later formulations of the PW model, it is assumed alternatively that

$$a_{21} = \frac{1}{k} p_k = \frac{c_0^2}{k}, \quad (6)$$

where c_0 is usually a fixed value. An important disadvantage of the PW models is, among others, their instability, especially at high traffic densities.

The PH (Phillips) model [5] takes additionally into account energy conservation. Its intention was to improve shortcomings of the PW models, among others by defining the pressure as a function of density

$$p = k \theta_0 \left(1 - \frac{k}{k_m} \right), \quad (7)$$

where θ_0 is a constant (chosen) parameter, and k_m is the maximum density. In view of the above, the quantity a_{21} now takes the form

$$a_{21} = \theta_0 \left(\frac{1}{k} - \frac{2}{k_m} \right), \quad (8)$$

and the quantity a_{22} remains unchanged (5). Unfortunately, it does not remove the drawbacks of the PW model and even causes new ones (such as acceleration at high densities).

The KK (Kerner–Konhäuser) model [6] is another attempt to improve the PW and PH models. In principle, it is based on inserting an additional viscosity to the traffic dynamics equation, which undoubtedly improves stability at low and high densities (although not at medium ones), and does not remove problems resulting from traffic isotropicity. From the perspective of this publication (neglecting the diffusive effects in model descriptions), the KK model does not add anything special to the PW model, and in relation to the PH model, it results in a constant sound speed.

The Z1 (Zhang, version 1) model [7] defines the speed of sound in the form

$$c_0 = ku_k, \quad (9)$$

which is equivalent to the assumption that the expression on pressure is

$$p = \frac{1}{3}k^3(u_k)^2, \quad (10)$$

and therefore the quantity a_{21} (3) is respectively

$$a_{21} = k(u_k)^2, \quad (11)$$

whereas the quantity a_{22} preserves the same form as in the PW and other models, mentioned above. The Z1 model removes some of the disadvantages of the PW model, because the quantity a_{21} is directly proportional to density (and not inversely proportional, as in the PW model). Nevertheless, many shortcomings are still relevant.

The MYL (Michalopoulos–Yi–Lyrintzis) model [8] can be defined using the following characteristic quantities:

$$p = \frac{\vartheta}{\gamma+2} k^{\gamma+2}, \quad (12)$$

$$a_{21} = \vartheta k^\gamma, \quad (13)$$

where ϑ is the anticipation parameter, and γ – a constant. This model does not use the velocity – density relationship, but refers to the free traffic speed. For the appropriate ϑ and γ , the MYL model takes the form of the Z1 model.

The AR (Aw–Rascle) model [9] is the first asymmetric (anisotropic) traffic model. This was mainly determined by the new form of defining the equation of traffic dynamics:

$$\partial_t(u + p) + u\partial_x(u + p) = 0, \quad (14)$$

but the pressure p has been determined differently than in other models. Multiplying the continuity equation (2) by $(-p_k)$ and adding it to the traffic equation (14) leads to the standard form of the AR model, in which the values a_{21} and a_{22} (3) are respectively:

$$a_{21} = 0 \quad a_{22} = u - kp_k. \quad (15)$$

Aw and Rascle [9] also suggested a pressure expression as

$$p = k^\gamma \quad (\gamma > 0), \quad (16)$$

which accordingly modifies the quantity a_{22} to the form

$$a_{22} = u - \gamma k^\gamma. \quad (17)$$

The advantages of the AR model (in relation to the models presented earlier) are unquestionable, but at very low densities, the solution may be unstable.

The Z2 (Zhang, version 2) model [10] does not relate the speed of sound c_0 to the entry a_{21} (3) (related to the spatial derivative of density), but to the entry a_{22} (related to the speed derivative). In this situation, both these quantities are respectively:

$$a_{21} = 0 \quad a_{22} = u + c_0, \quad (18)$$

where the speed of sound c_0 is defined as follows:

$$c_0 = ku_k \quad (u_k < 0). \quad (19)$$

As can be seen, the models AR and Z2 are very closely related – the difference stems only from the determination of derivatives p_k and u_k respectively on the basis of various equilibrium relations, flow – density or speed – density.

The JWZ (Jiang-Wu-Zhu) model [11], similarly to the Z2 model, describes the anticipation using the speed gradient, not the density gradient. As a result, the quantity a_{21} and a_{22} (3) are defined here a bit similar to relationships (18), namely:

$$a_{21} = 0 \quad a_{22} = u - c_0. \quad (20)$$

Assuming according to (15) that the speed of sound is $c_0 = kp_k$, then, on this basis, it becomes possible to determine the pressure p as

$$p = c_0 \ln k. \quad (21)$$

The properties of the JWZ model are similar to the AR and Z2 models.

The above concise presentation of macroscopic traffic models has been limited to the most popular models that formed the basis for further development of their specific variants. Collected and outlined (in the above paragraphs) characteristic features of the mathematical description of models and the resulting properties of solutions, provide a sufficient basis to compare the conditioning of various formulations for particular classes and subclasses of traffic models, and moreover, to assess their quality and efficiency resulting only from their formulations on a continuous level. This means that discretization and approximation of models will not be taken into account.

3. Considerations on the conditioning of traffic models

Considerations on the conditioning of different formulations of macroscopic road traffic models will be carried out first in a general approach, i.e. in relation to the matrix formulation (3), where specific, different quantities a_{21} and a_{22} correspond to the formulation of individual traffic models. After consideration on the generalised formulation and defining the measures of conditioning for models, further considerations will include references to particular classes of traffic models.

The characteristic polynomial of the general formulation of traffic models (3) has the form

$$\begin{vmatrix} u - \lambda & k \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (u + a_{22})\lambda + ua_{22} - \rho a_{21} = 0, \quad (22)$$

and its solutions to λ (eigenvalues) are:

$$\lambda_{1,2} = \frac{1}{2} \left(u + a_{22} \pm \sqrt{(u - a_{22})^2 + 4ka_{21}} \right). \quad (23)$$

It is extremely important in the context of later considerations to express the values a_{21} and a_{22} as a function of eigenvalues λ_1 and λ_2 , which can be described by the following relationships:

$$a_{21} = \frac{1}{\rho} (\lambda_1 - u)(u - \lambda_2) \quad (24)$$

$$a_{22} = \lambda_1 + \lambda_2 - u. \quad (25)$$

In order to compare the conditioning of macroscopic traffic models, carried out at the level of mathematical formulations, i.e. without taking into account the methods of discretization and

approximation of the problem, and without reference to the quality of numerical development and computer implementation, the following measures were assumed:

- the spectral radius,
- the spectral condition number.

The spectral radius ρ_S of the matrix system of model equations is defined by the largest λ value of this system. Thus, it is defined as follows:

$$\rho_S = \max(\lambda_1, \lambda_2), \quad (26)$$

where eigenvalues λ_1 and λ_2 are generally defined by relationship (23). If all the considered traffic speeds are positive ($u > 0$), then the spectral radius is then

$$\rho_S = \frac{1}{2} \left(u + a_{22} + \sqrt{(u - a_{22})^2 + 4ka_{21}} \right). \quad (27)$$

In accordance with the above definition of the ρ_S radius, it is now possible to refer more specifically to traffic models. In order not to analyse all of the models presented earlier, only two main classes of traffic models are considered further:

- symmetrical (isotropic) models, including models: PE, PH, KK, Z1 and MYL, and
- asymmetric (anisotropic) models, containing models: AR, Z2, and JWZ.

Isotropic models are characterised by eigenvalues generally referred to as

$$\lambda_1 = u + c_0 \quad \lambda_2 = u - c_0, \quad (28)$$

where c_0 is the speed of sound specific for a given symmetrical model. Insertion of relationships (28) into (24) and (25) results in the following expressions on a_{21} and a_{22} :

$$a_{21} = \frac{1}{k} c_0^2 \quad a_{22} = u. \quad (29)$$

In such a situation, the spectral radius of the symmetric (isotropic) models is always

$$\rho_{SS} = u + c_0, \quad (30)$$

where individual isotropic models are characterised by their own speeds of sound c_0 . It can therefore be concluded that the lower the speed c_0 for the given isotropic model, the better it is conditioned.

Anisotropic models assume that the highest wave speed λ_1 cannot be greater than the driving speed – in practice, it is simply assumed that

$$\lambda_1 = u. \quad (31)$$

Inserting the above relationship into (24) and (25) allows defining the quantities a_{21} and a_{22} as

$$a_{21} = 0 \quad a_{22} = \lambda_2. \quad (32)$$

This means that the spectral radius of asymmetric (anisotropic) models is in such a situation

$$\rho_{SA} = u, \quad (33)$$

and thereby it is equal for all anisotropic models, and therefore the conditioning of these models is the same.

The condition number κ of the coefficient matrix of the system of equations is defined in general as the product of the matrix norm and the inverse matrix norm, which indicates the dependence of the condition number on the adopted matrix norm. If all of the eigenvalues of the matrix are real, then for the induced matrix norm $\|\cdot\|_2$, the condition number assumes the general form

$$\kappa = \frac{|\lambda_{max}|}{|\lambda_{min}|} \quad (34)$$

and is referred to as the spectral condition number. In respect of studied systems of equations having two eigenvalues, where also the traffic speeds are positive ($u > 0$), the spectral condition number can be defined as

$$\kappa = \max\left(\frac{|\lambda_1|}{|\lambda_2|}, \frac{|\lambda_2|}{|\lambda_1|}\right) = \frac{\lambda_1}{|\lambda_2|} \quad (35)$$

or, after considering the relationship (23), as

$$\kappa = \frac{u+a_{22}+\sqrt{(u-a_{22})^2+4ka_{21}}}{u+a_{22}-\sqrt{(u-a_{22})^2+4ka_{21}}}. \quad (36)$$

In respect of symmetric (isotropic) models characterised by the eigenvalues (28), and therefore also relationships (29), the condition number is

$$\kappa_S = \frac{u+c_0}{|u-c_0|}, \quad (37)$$

where the speed of sound c_0 is characteristic for a specific model. This means that the smaller the speed c_0 related to the specific isotropic model, the better it is conditioned.

In the case of asymmetric (anisotropic) models, where the highest wave speed corresponds to the traffic speed (31), and consequently also the relationships (32) are relevant, the condition number is

$$\kappa_A = \frac{u}{|a_{22}|}. \quad (38)$$

Since in general the quantity a_{22} for the positive c_0 is described by the relationship (20), and for the negative c_0 – by (18), then assuming the speed of sound c_0 as a positive value, the condition number is determined as follows:

$$\kappa_A = \frac{u}{|u-c_0|}. \quad (39)$$

It follows clearly that anisotropic models characterised by a lower velocity c_0 are better conditioned than models with a higher speed of sound.

4. Conclusions

Conditioning measures determined for various macroscopic traffic models, were the basis for the following assessments and conclusions:

1. The spectral radius of symmetric (isotropic) traffic models (30) depends on the propagation speed of traffic disturbances (the speed of sound). The smaller it is, the better conditioned the model is.
2. The spectral radius of asymmetric (anisotropic) traffic models (33) is equal for all models, so their conditioning is identical.
3. All anisotropic traffic models are better conditioned (in terms of spectral radiuses) than isotropic models.
4. The condition number for symmetric traffic models (37) depends on the speed of sound - the higher it is, the worse the conditioning is.
5. The condition number for asymmetric traffic models (39) also depends on the speed of sound; however, this relationship is different than for the isotropic models. Nevertheless, the principle still applies that increasing the speed of sound worsens the conditioning.
6. All anisotropic traffic models are characterised by a better conditioning (in terms of the condition numbers) than isotropic models.
7. In terms of the level of conditioning, it is possible to compare not only classes or subclasses of macroscopic traffic models, but also any individual traffic models. However, it should be remembered (especially when assessing according to the condition number) that model conditioning may change with the change in the ratio of the speed of sound to the speed of traffic (or vice versa, the speed of traffic to the speed of sound).

Acknowledgments

The article was financed from the funds of the project no. 05/51/DS-PB/3520.

References

- [1] Lighthill M J and Whitham G B 1955 On kinematic waves: II. A theory of traffic flow on long crowded roads *Proceedings of the Royal Society A* **229** (London: Royal Society) pp 317–345
- [2] Richards P I 1956 Shockwaves on the highway *Operations Researches* **4** pp 42–51
- [3] Payne H J 1971 Models of freeway traffic and control *Mathematical Models of Public Systems* **1** (La Jolla: Simulation Councils) pp 51–61
- [4] Whitham G B 1974 *Linear and nonlinear waves* (New York: John Wiley and Sons)
- [5] Phillips W F 1979 A kinetic model for traffic flow with continuum implications *Transportation Planning and Technology* **5** pp 131–138
- [6] Kerner B S and Konhäuser P 1993 Cluster effect in initially homogeneous traffic flow *Physical Review E* **48** pp 2335–2338
- [7] Zhang H M 1998 A theory of nonequilibrium traffic flow *Transportation Research B* **32** pp 485–498
- [8] Michalopoulos P G, Yi P and Lyrantzis A S 1993 Continuum modeling of traffic dynamics for congested freeways *Transportation Research B* **27** pp 315–332
- [9] Aw A and Rascle M 2000 Resurrection of second order models of traffic flow *SIAM Journal of Applied Mathematics* **60** pp 916–938
- [10] Zhang H M 2002 A non-equilibrium traffic model devoid of gas-like behavior *Transportation Research B* **36** pp 275–290
- [11] Jiang R, Wu Q-S and Zhu Z-J 2002 A new continuum model for traffic flow and numerical tests *Transportation Research B* **36** pp 405–419