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Electric and elastic singularity eigenanalysis of the V-notch in a piezoelectric Reissner plate

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Abstract. The singular eigenstate with respect to the stress and electric displacement at the vertex of a V-notch in a piezoelectric plate is investigated. The displacement and electric potential near the vertex of a V-notch are expressed as series asymptotic expansions, and their typical terms are substituted into the governing equations to establish the singularity eigenequations, from which the singularity orders and angular eigenfunctions can be yielded by the interpolating matrix method. The coupling effect of elastic and electric singularities can be found by distinguishing the singularity orders from their angular eigenfunctions. There are most four singularity orders for the V-notch in a piezoelectric Reissner plate, which is one more than the one in a homogeneous plate. Two orders are corresponding to the singularity of bending moment and electric displacement in the thickness direction, and another two orders are with respect to the singularity of shear force and electric displacement in the mid-plane. The free and electrically opened, and clamped and electrically closed boundary conditions are respectively introduced to investigate the influence of boundary conditions on the singularity. It is found that the singularity orders with respect to bending moment and electric displacement in the thickness direction change with boundary conditions, while the singularity orders with respect to shear force and electric displacement in the mid-plane are identical under these two kinds of boundary conditions.

1. Introduction

Due to the direct-converse piezoelectric effect, the piezoelectric material has been extensively used in electromechanical transducers, sensors, etc. [1]. However, these piezoelectric structures or components always include V-notches, which are generated by the discontinuity of material property and/or geometry. There will present the singularities near the vertex of a V-notch, in which the stress singularity may lead to the crack initiation and extension, and the singularity of the electric displacement may lead to the electrical breakdown.

As the specific case of a V-notch, the singularity eigenanalysis for the crack in a piezoelectric structure had been thoroughly investigated [2-4]. A piezoelectric structure in the energized state will produce the heat, which will make the thermal stress singularity near the crack tip and weaken the structures. The fracture problem for cracks in piezoelectric material subjected to thermal loading attracted much attention [5-7]. In comparison with the crack problem, the work on the singularity analysis for a piezoelectric notch is much less [8-10]. The piezoelectric structures are often used in a plate or shell model [11,12]. According to the best knowledge of authors, the present research is mainly focused on the piezoelectric V-notch in plane, anti-plane or three-dimensional state [13]. There are scarce data on the characteristic singularity analysis for the V-notch in a piezoelectric plate V-notch. The difficulty is from the coupling of the stress singularity and electric displacement singularity.

The motivation of the present paper is to propose the singularity eigenanalysis of stress and electric displacement for a piezoelectric plate V-notch. The elastic and electrical equilibrium equations for a piezoelectric Reissner plate are firstly given out. Then, the typical terms in the asymptotic displacement and potential electric fields near the singular vertex are substituted to the governing equations together with radial boundary conditions to establish singularity eigen equations, which will be solved by the interpolating matrix method [14] to yield singularity orders and angular eigenfunctions. The coupled effect of the elastic singularity and electric singularity is investigated.

2. Equilibrium equations for a piezoelectric plate notch

Let's consider a piezoelectric plate weakened by a single edge V-notch. The electric potential is assumed to vary as a parabolic function along the thickness direction [15], according to the following equation

$$\psi(r,\theta,z)=\frac{h^2}{4-z^2}\psi_0(r,\theta) \tag{1}$$

where ψ and ψ_0 are the electric potentials in the plate and mid-plane, respectively, while h is the plate thickness. The equilibrium differential equations regarding generalized displacement components for a piezoelectric Reissner plate without body force can be written as follows

$$\begin{aligned}&D(\frac{\partial^2\phi_r}{\partial r^2}+\frac{\partial\phi_r}{r\partial r}-\frac{\phi_r}{r^2}+\frac{1-\nu}{2r^2}\frac{\partial^2\phi_r}{\partial\theta^2}+\frac{1+\nu}{2r}\frac{\partial^2\phi_\theta}{\partial r\partial\theta}-\frac{3-\nu}{2r^2}\frac{\partial\phi_\theta}{\partial\theta^2})+C(\frac{\partial w}{\partial r}+\frac{d_{15}h^2}{5}\frac{\partial\psi_0}{\partial r}-\phi_r)\\&+\frac{d_{31}h^3}{6s_{11}(1-\nu)}\frac{\partial\psi_0}{\partial r}=0\end{aligned} \tag{2a}$$

$$\begin{aligned}&D(\frac{1+\nu}{2r}\frac{\partial^2\phi_r}{\partial r\partial\theta}+\frac{3-\nu}{2r^2}\frac{\partial\phi_r}{\partial\theta}+\frac{1-\nu}{2}\frac{\partial^2\phi_\theta}{\partial r^2}+\frac{1-\nu}{2r}\frac{\partial\phi_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi_\theta}{\partial\theta^2}-\frac{1-\nu}{2}\frac{\phi_\theta}{r^2})\\&+C(\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{d_{15}h^2}{5r}\frac{\partial\psi_0}{\partial\theta}-\phi_\theta)+\frac{d_{31}h^3}{6rs_{11}(1-\nu)}\frac{\partial\psi_0}{\partial\theta}=0\end{aligned} \tag{2b}$$

$$\frac{\partial^2 w}{\partial r^2}+\frac{\partial w}{r\partial r}+\frac{\partial^2 w}{r^2\partial\theta^2}+\frac{d_{15}h^2}{5}(\frac{\partial^2\psi_0}{\partial r^2}+\frac{\partial\psi_0}{r\partial r}+\frac{\partial^2\psi_0}{r^2\partial\theta^2})-(\frac{\partial\phi_r}{\partial r}+\frac{\phi_r}{r}+\frac{\partial\phi_\theta}{r\partial\theta})=0 \tag{2c}$$

$$-5(\frac{d_{31}}{s_{11}+s_{12}}+\frac{d_{15}}{s_{44}})(\frac{\partial\phi_r}{\partial r}+\frac{\phi_r}{r}+\frac{\partial\phi_\theta}{r\partial\theta})+\frac{5d_{15}}{s_{44}}\nabla^2 w+h^2(\frac{d_{15}^2}{s_{44}}-\beta_{11})\nabla^2\psi_0 \\ +10[\frac{2d_{31}^2}{s_{11}(\nu-1)}+\beta_{33}]\psi_0=0 \tag{2d}$$

where $\nu=-s_{12}/s_{11}$, $D=\frac{h^3}{12(1-\nu^2)s_{11}}$, $C=\frac{5h}{6s_{44}}$, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, ϕ_r and ϕ_θ are the bending rotations of the mid-plane normal in radial and circumferential directions, respectively, w is the displacement component along the z-direction. s_{11}, s_{12}, s_{44} are flexibility coefficients, d_{15} and d_{31} are piezoelectric coefficients, β_{11} and β_{33} are permittivity values. The piezoelectric material PZT-4 with a z-axial polarization is taken into consideration, with the following material properties [8]:

$s_{11}=10.90\times10^{-12}\,\mathrm{m^2N^{-1}}$, $s_{12}=-5.42\times10^{-12}\,\mathrm{m^2N^{-1}}$, $s_{44}=19.30\times10^{-12}\,\mathrm{m^2N^{-1}}$, $d_{15}=39.40\times10^{-3}\,\mathrm{VmN^{-1}}$,

$d_{31}=-11.10\times10^{-3}\,\mathrm{VmN^{-1}}$, $\beta_{11}=7.66\times10^{7}\,\mathrm{V^2N^{-1}}$, $\beta_{33}=8.69\times10^{7}\,\mathrm{V^2N^{-1}}$.

If two radial edges of the V-notch are supposed to be set traction-free and electrically opened, this boundary condition can be reduced to the following form

$$M_\theta=0,\ M_{r\theta}=0,\ Q_\theta=0,\ \int_{-\frac{h}{2}}^{\frac{h}{2}}D_\theta(h^2/4-z^2)\mathrm{d}z=0\quad(\theta=-\alpha,\alpha) \tag{3}$$

where α is half of the notch opening angle. These can be represented by the generalized displacement components $(\phi_r,\phi_\theta,w,\psi_0)$,

$$\frac{1}{6(s_{11}-s_{12})}\left[s_{12}\frac{\partial\phi_r}{\partial r}-s_{11}(\frac{1}{r}\frac{\partial\phi_\theta}{\partial\theta}+\frac{1}{r}\phi_r)\right]-d_{31}\psi_0=0\quad(\theta=-\alpha,\alpha) \tag{4a}$$

$$\frac{1}{r}\frac{\partial\phi_r}{\partial\theta}+\frac{\partial\phi_\theta}{\partial r}-\frac{\phi_\theta}{r}=0\quad(\theta=-\alpha,\alpha) \tag{4b}$$

$$5(-\phi_\theta+\frac{1}{r}\frac{\partial w}{\partial\theta})+h^2d_{15}\frac{\partial\psi_0}{r\partial\theta}=0\quad(\theta=-\alpha,\alpha) \tag{4c}$$

$$\beta_{11}\frac{1}{r}\frac{\partial\psi_0}{\partial\theta}-\frac{5d_{15}}{s_{44}h^2}(-\phi_\theta+\frac{\partial w}{r\partial\theta})-\frac{d_{15}^2}{s_{44}}\frac{1}{r}\frac{\partial\psi_0}{\partial\theta}=0\quad(\theta=-\alpha,\alpha) \tag{4d}$$

If two radial edges of the V-notch are supposed to be set clamped and electrically closed, this kind of boundary conditions can be written as

$$\phi_r=0,\ \phi_\theta=0,\ w=0,\ \psi_0=0\quad(\theta=-\alpha,\alpha) \tag{5}$$

3. Singularity eigen equations for a piezoelectric plate notch

The elastic and electric displacement field near the vertex of a V-notch can be expressed by a series of asymptotic expansions as follows [16]

$$\phi_r=\sum_{k=1}^{N}A_k r^{\lambda_k}\tilde{\phi}_{rk}(\lambda_k,\theta),\ \phi_\theta=\sum_{k=1}^{N}A_k r^{\lambda_k}\tilde{\phi}_{\theta k}(\lambda_k,\theta),\ w=\sum_{k=1}^{N}A_k r^{\lambda_k}\tilde{w}_k(\lambda_k,\theta),\ \psi_0=\sum_{k=1}^{N}A_k r^{\lambda_k}\tilde{\psi}_{0k}(\lambda_k,\theta) \tag{6}$$

where r is the radial distance to the notch tip, A_k is the amplitude coefficient, λ_k is the singularity order, N is the number of truncated series terms, $\tilde{\phi}_{rk}(\lambda_k,\theta)$, $\tilde{\phi}_{\theta k}(\lambda_k,\theta)$, $\tilde{w}_k(\lambda_k,\theta)$ and $\tilde{\psi}_{0k}(\lambda_k,\theta)$ are angular eigenfunctions of the generalized displacements.

Substituting the typical terms in Equation (6), i.e., $A_k r^{\lambda_k}\tilde{\phi}_{rk}(\lambda_k,\theta)$, $A_k r^{\lambda_k}\tilde{\phi}_{\theta k}(\lambda_k,\theta)$, $A_k r^{\lambda_k}\tilde{w}_k(\lambda_k,\theta)$ and $A_k r^{\lambda_k}\tilde{\psi}_{0k}(\lambda_k,\theta)$ into Equation (2) and letting the sum of the coefficients of the lowest order of r set as zero, yields the ordinary differential eigenequations for the V-notch in a piezoelectric plate, which can be written as

$$\frac{1-\nu}{2}\tilde{\phi}_r''-\frac{3-\nu}{2}\tilde{\phi}_\theta'-\tilde{\phi}_r+\lambda\frac{1+\nu}{2}\tilde{\phi}_\theta'+\lambda^2\tilde{\phi}_r=0 \tag{7a}$$

$$\tilde{\phi}_\theta''+\frac{3-\nu}{2}\tilde{\phi}_r'-\frac{1-\nu}{2}\tilde{\phi}_\theta+\lambda\frac{1+\nu}{2}\tilde{\phi}_r'+\lambda^2\frac{1-\nu}{2}\tilde{\phi}_\theta=0 \tag{7b}$$

$$\tilde{w}''+\frac{h^2 d_{15}}{5}\tilde{\psi}_0''+\lambda^2 w+\lambda^2\frac{h^2 d_{15}}{5}\tilde{\psi}_0=0 \tag{7c}$$

$$\tilde{w}''+\frac{h^2}{5}(d_{15}-\frac{s_{44}\beta_{11}}{d_{15}})\tilde{\psi}_0''+\lambda^2 w+\lambda^2\frac{h^2}{5}(d_{15}-\frac{s_{44}\beta_{11}}{d_{15}})\tilde{\psi}_0=0 \tag{7d}$$

where $(\cdots)'=\mathrm{d}(\cdots)/\mathrm{d}\theta$, $(\cdots)''=\mathrm{d}^2(\cdots)/\mathrm{d}\theta^2$, λ, $\tilde{\phi}_r$, $\tilde{\phi}_\theta$, $\tilde{w}$ and $\tilde{\psi}_0$ are the simplified forms of λ_k, $\tilde{\phi}_{rk}(\lambda_k,\theta)$, $\tilde{\phi}_{\theta k}(\lambda_k,\theta)$, $\tilde{w}_k(\lambda_k,\theta)$ and $\tilde{\psi}_{0k}(\lambda_k,\theta)$, respectively.

Similarly introducing the typical terms from Equation (6) into Equation (4), the traction-free and electrically opened boundary conditions can be expressed by angular eigenfunctions and their derivatives like

$$\tilde{\phi}_\theta'+\tilde{\phi}_r-\frac{s_{12}}{s_{11}}\lambda\tilde{\phi}_r=0,\quad \tilde{\phi}_r'-\tilde{\phi}_\theta+\lambda\tilde{\phi}_\theta=0,\quad \tilde{w}'+\frac{h^2 d_{15}}{5}\tilde{\psi}_0'=0,\quad \tilde{w}'+\frac{h^2}{5}(d_{15}-\frac{s_{44}\beta_{11}}{d_{15}})\tilde{\psi}_0'=0 \tag{8}$$

The clamped and electrically closed boundary condition can be rewritten as

$$\tilde{\phi}_r=0,\quad \tilde{\phi}_\theta=0,\quad \tilde{w}=0,\quad \tilde{\psi}_0=0\quad (\theta=-\alpha,\alpha) \tag{9}$$

Until now, the singularity analysis for the V-notch in a piezoelectric Reissner plate was reduced to solve the singularity eigenequation (7) under radial boundary conditions described by Equations (8) or (9). The interpolating matrix method, which was earlier proposed in [14] and used by some authors, is employed here to solve the singularity eigenequations (7, 8) or Equations (7, 9), from which the singular orders together with angular eigenfunctions for the V-notch in a piezoelectric plate can be derived.

4. Numerical examples and discussions

The influence of plate thickness on the singularity of the V-notch in a piezoelectric plate is firstly investigated. There are three different plate thicknesses (i.e., 0.001, 0.01, and 0.1m), taken into consideration. The variation of singular orders with plate thickness for the V-notch with $2\alpha=300^\circ$ is listed in table 1, where traction-free and electrically opened boundary conditions are considered. It can be seen that the singularity orders do not vary dramatically with the plate thickness. Thus, the plate thickness is set at $h = 0.1\text{m}$ in the further evaluation.

Table 1. Variation of singularity orders with plate thickness for $2\alpha = 300°$.

h(m)	λ_1	λ_2	λ_3	λ_4
0.001	0.512225	0.600000	0.600000	0.730924
0.010	0.512223	0.600000	0.600000	0.730912
0.100	0.512223	0.600000	0.600000	0.730912

The calculated singularity orders for V-notches with different opening angles are listed in table 2, where traction-free and electrically opened boundary conditions is adopted. It can be seen that there are three singularity orders for $180° < 2\alpha \le 260°$, and four singularity orders for $260° < 2\alpha \le 360°$. The singularity orders corresponding to bending moment, shear force and electric displacement can be distinguished by the angular eigenfunctions.

Table 2. Singularity orders for V-notches under traction-free and electrically opened boundary conditions.

2α(°)	Method	λ_1	λ_2	λ_3	λ_4
180	Present	/	/	/	/
	Ref.[17]	/	/	/	/
220	Present	0.697167	0.818182	0.818182	/
	Ref.[17]	0.697165	0.818182	-	
260	Present	0.562841	0.692308	0.692308	0.980489
	Ref.[17]	0.562839	0.692308	-	0.980474
300	Present	0.512223	0.600000	0.600000	0.730912
	Ref.[17]	0.512221	0.600000	-	0.730901
340	Present	0.500429	0.529412	0.529412	0.562017
	Ref.[17]	0.500426	0.529412	-	0.562007
360	Present	0.500000	0.500000	0.500003	0.500009
	Ref.[17]	0.500000	0.500000	-	0.500000

Note: the slash means no singularity and the dash means no corresponding item in the reference.

The singularity orders for the V-notch in a homogeneous plate by the eigen function expansion method [17] are listed in table 2 to provide the reference ones. It can be found out that the present results are in good agreement with the reference ones for λ_1, λ_2 and λ_4, except for λ_3, which was not provided in 17]. Thus, one singularity order will be increased for the piezoelectric plate notch against the homogeneous plate notch.

The calculated singularity orders for the V-notches under clamped and electrically closed boundary condition are listed in table 3, where the results from [17] are also listed for reference. It should be pointed out that, the singularity orders do not depend on Poisson's ratio for the V-notch under free boundary condition, while they depend on Poisson's ratio under clamped boundary condition for the V-notch in a homogeneous Reissner plate. Thus, the reference results are calculated from the V-notch with Poisson's ratio $\nu = -s_{12}/s_{11}$. From table 3, it can be seen that the present results agree well with the reference ones, and one more singularity order is derived for the V-notch in a piezoelectric plate, as compared to the V-notch in a homogeneous plate.

Table 3. Singularity orders for V-notches under clamped and electrically closed boundary condition.

2α(°)	Method	λ_1	λ_2	λ_3	λ_4
180	Present	/	/	/	/
	[17]	/	/	/	/
220	Present	0.742749	0.818182	0.818182	0.911443
	[17]	0.742747	0.818182	-	0.911441
260	Present	0.611143	0.692308	0.692308	0.800546
	[17]	0.611140	0.692308	-	0.800542
300	Present	0.545301	0.600000	0.600000	0.667441
	[17]	0.545299	0.600000	-	0.667434
340	Present	0.511738	0.529412	0.529412	0.548367
	[17]	0.511735	0.529412	-	0.548359
360	Present	0.500000	0.500000	0.500004	0.500008
	[17]	0.500000	0.500000	-	0.500000

Note: the slash means no singularity and the dash means no corresponding item in the reference.

It had been indicated in [17] that the singularity order with respect to shear force keeps the same for the V-notch under free boundary condition and clamped boundary condition. By comparing tables 2 and 3, it can be found that the singularity orders λ_2 and λ_3 are identical in these two tables, which means that the singularity orders concerning shear force and electric displacement in the mid-plane keep identical under free and electrically opened boundary condition, and clamped and electrically closed boundary condition. However, λ_1 and λ_4 are different in these two kinds of boundary conditions.

5. Conclusions

The singular orders for the V-notch in a piezoelectric Reissner plate were investigated. It was found that the number of singularity orders in a piezoelectric plate notch exceeds by one that in a homogeneous plate notch. The coupling effect of elastic singularity and electric singularity in a piezoelectric plate V-notch was found. To investigate the influence of the boundary conditions on the singularity of the V-notch, the free and electrically opened, and clamped and electrically closed boundary conditions were taken into consideration. Under these two different boundary conditions, the singularity orders with respect to bending moment and electric displacement in the thickness direction are different, while the singularity orders with respect to shear force and electric displacement in the mid-plane remain identical.

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